The Phenomenon of ℓ-independence

Galois representations, companions and monodromy over finite fields

Suvir Rathore

Galois groups acting on invariants

SETTING

Let $K = \mathbb{Q}$ or $\mathbb{F}_p$, fix an algebraic closure $\bar{K}$ and write $G_K := \mathrm{Gal}(\bar{K}/K)$. A variety X over K comes with an action of G_K on $X(\bar{K})$, and this action is inherited by every “nice” invariant attached to X (torsion points, cohomology groups, ...). These invariants are the objects whose dependence on an auxiliary prime ℓ we want to understand.

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EXAMPLE 1 — THE MULTIPLICATIVE GROUP

$X = \mathbb{G}_m$, the punctured affine line: $\mathbb{G}_m(\bar{K}) = \bar{K}^\times$ with its G_K-action. Fix a prime ℓ ($\ell \neq p$ if $K = \mathbb{F}_p$). For each $n \geq 1$ the roots of unity of order dividing ℓ^n form a G_K-stable subgroup, and these are packaged by an inverse limit (transition maps $\zeta \mapsto \zeta^\ell$):

$$\mu_{\ell^n}(\bar{K}) \;\cong\; \mathbb{Z}/\ell^n\mathbb{Z}, \qquad T_\ell\mathbb{G}_m := \varprojlim_n \mu_{\ell^n}(\bar{K}) \;\cong\; \mathbb{Z}_\ell, \qquad V_\ell\mathbb{G}_m := T_\ell\mathbb{G}_m \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

So $V_\ell\mathbb{G}_m$ is a 1-dimensional continuous representation of G_K over $\mathbb{Q}_\ell$: the cyclotomic character

$$\chi_\ell : \; G_K \longrightarrow \mathbb{Z}_\ell^\times \subset \mathbb{Q}_\ell^\times, \qquad \sigma(\zeta) = \zeta^{\chi_\ell(\sigma)} \quad \text{for all } \zeta \in \mu_{\ell^\infty}(\bar{K}).$$

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REMARK

For $K = \mathbb{F}_p$ one must take $\ell \neq p$: $\mu_{p^n}(\bar{\mathbb{F}}_p) = \{1\}$, so $T_p\mathbb{G}_m = 0$. The case "$\ell = p$" is the territory of crystalline and p-adic methods, which reappear below.

Frobenius does not see ℓ

EXAMPLE 1, CONTINUED — FROBENIUS ON THE CYCLOTOMIC CHARACTER

$K = \mathbb{F}_q$, $q = p^f$. Then $G_K \cong \widehat{\mathbb{Z}}$ is topologically generated by the arithmetic Frobenius $\sigma_q : x \mapsto x^q$, which acts on μ_{ℓ^n} by $\zeta \mapsto \zeta^q$. Hence $\chi_\ell(\sigma_q) = q$ and

$$\det\left(T - \sigma_q \mid V_\ell \mathbb{G}_m\right) = T - q \in \mathbb{Z}[T] \qquad \text{is independent of } \ell \neq p.$$

Likewise for $K = \mathbb{Q}$: for a prime $p \neq \ell$ the (unramified) Frobenius Frob_p acts by $\zeta \mapsto \zeta^p$, so $\chi_\ell(\mathrm{Frob}_p) = p$ for every $\ell \neq p$.

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EXAMPLE 2 — AN ELLIPTIC CURVE OVER A FINITE FIELD

E an elliptic curve over $\mathbb{F}_q$, $\ell \neq p$. Then $E[\ell^n](\bar{\mathbb{F}}_q) \cong (\mathbb{Z}/\ell^n\mathbb{Z})^2$, and the ℓ-adic Tate module

$$T_\ell E := \varprojlim_n E[\ell^n](\bar{\mathbb{F}}_q) \ \cong\ \mathbb{Z}_\ell^2, \qquad V_\ell E := T_\ell E \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is a 2-dimensional representation of G_K, on which σ_q acts through the Frobenius endomorphism $\pi \in \mathrm{End}(E)$. For every $\varphi \in \mathrm{End}(E)$ one has $\det(\varphi \,|\, T_\ell E) = \deg \varphi$ and $\mathrm{tr}(\varphi \,|\, T_\ell E) = 1 + \deg \varphi - \deg(1 - \varphi)$, so

$$\det\bigl(T - \sigma_q \bigm| V_\ell E\bigr) = T^2 - a_q T + q, \qquad a_q := q + 1 - \#E(\mathbb{F}_q) \in \mathbb{Z}, \qquad \text{independent of } \ell.$$

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TATE MODULE VERSUS ℓ-ADIC COHOMOLOGY

Tate modules are dual to cohomology: $H^1_{\text{ét}}(E_{\bar{\mathbb{F}}_q}, \mathbb{Z}_\ell) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell E, \mathbb{Z}_\ell)$, and via the Weil pairing this is $T_\ell E(-1)$. The geometric Frobenius $F = \sigma_q^{-1}$ on H^1 has the same characteristic polynomial $T^2 - a_q T + q$, and Grothendieck's trace formula reads $\#E(\mathbb{F}_q) = 1 - a_q + q$: ℓ-independence of Frobenius on ℓ-adic cohomology, in the first non-trivial case.

One object, many realisations

GROTHENDIECK'S PROGRAMME

Grothendieck's dream. There should be a $\mathbb{Q}$-linear semisimple Tannakian category $\mathrm{Mot}(K)$ of pure motives and a universal cohomology

$$h : \{\text{smooth projective } K\text{-varieties}\}^{\mathrm{op}} \longrightarrow \mathrm{Mot}(K), \qquad X \longmapsto h(X) = \textstyle\bigoplus_i h^i(X),$$

through which every Weil cohomology theory factors via a realisation functor.
Evidence: the comparison isomorphisms

- Artin ($K \subset \mathbb{C}$): $H^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ for every prime ℓ;
- de Rham: $H^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i_{\mathrm{dR}}(X/K) \otimes_K \mathbb{C}$;
- Katz–Messing ($K = \mathbb{F}_q$): Frobenius on $H^i_{\mathrm{cris}}(X/W) \otimes \mathbb{Q}$ and on $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ have the same characteristic polynomial.

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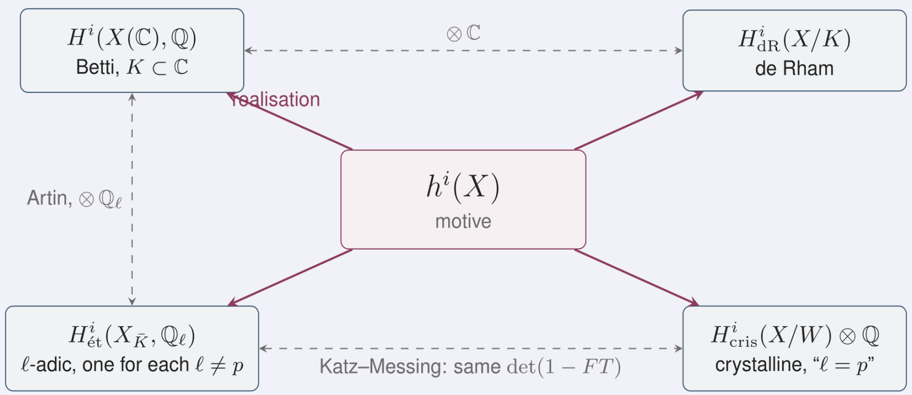
THE REALISATION FUNCTORS



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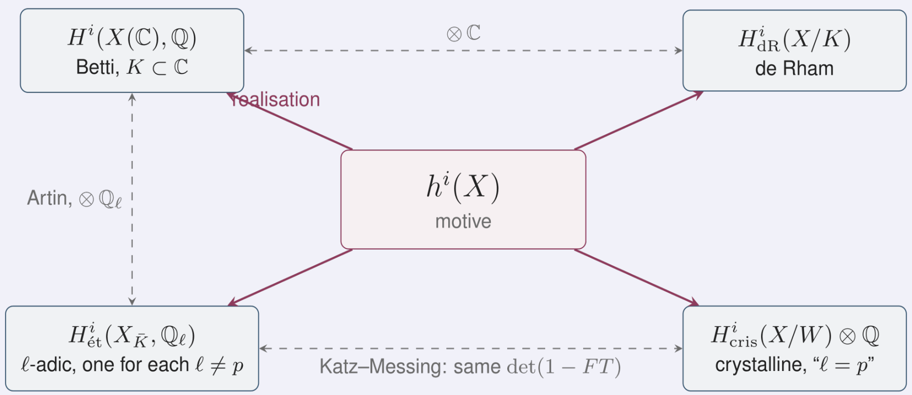

WHAT ℓ-INDEPENDENCE SHOULD MEAN

Expectation. The ℓ-adic realisations $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$, $\ell \neq p$, are all shadows of the one object $h^i(X)$, so whatever is "motivic" should not depend on ℓ. For $K = \mathbb{F}_q$: $\det(1 - FT \mid H^i_{\text{ét}})$ should lie in $\mathbb{Z}[T]$, and the Zariski closure of the image of G_K in $\mathrm{GL}(H^i_{\text{ét}})$ should be the ℓ-adic realisation of one motivic Galois group. For K a number field, primes of good reduction reduce to this by smooth proper base change; at bad primes one works with Weil–Deligne representations, and their ℓ-independence is known for H^1 (Grothendieck) but open in general.

Frobenius polynomials are integers, independent of ℓ

THEOREM (WEIL CONJECTURES: DWORK, GROTHENDIECK, DELIGNE)

Let X be smooth projective of dimension d over $\mathbb{F}_q$ and $Z(X,T) := \exp\Big(\sum_{n\geq 1} \#X(\mathbb{F}_{q^n})\,\frac{T^n}{n}\Big)$. For every prime $\ell \neq p$:

(1) *Rationality* (Dwork 1960; Grothendieck via the Lefschetz trace formula):

$$Z(X,T) = \prod_{i=0}^{2d} P_i(T)^{(-1)^{i+1}}, \qquad P_i(T) := \det\big(1 - FT \mid H^i_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)\big), \quad F = \text{geometric Frobenius.}$$

(2) *Functional equation* (Poincaré duality): $Z(X, 1/q^d T) = \pm\, q^{d\chi/2}\, T^{\chi}\, Z(X,T)$, where $\chi = \sum_i (-1)^i \dim H^i_{\text{ét}}$.

(3) *Riemann hypothesis* (Deligne 1974): $P_i(T) = \prod_j (1 - \alpha_{ij} T)$ with α_{ij} algebraic integers such that $|\iota(\alpha_{ij})| = q^{i/2}$ for every embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

(4) If X is the reduction of a smooth projective variety Y over a number field, then $\deg P_i = \dim H^i(Y(\mathbb{C}), \mathbb{Q})$.

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CONSEQUENCE

ℓ-independence. $Z(X,T)$ is defined by point counts, so it does not know ℓ. By (3) the inverse roots of P_i have absolute value $q^{i/2}$, so no cancellation between different i can occur and each P_i is determined by $Z(X,T)$ alone; its coefficients are rational (Galois-stable factorisation) and algebraic integers. Hence $P_i(T) \in \mathbb{Z}[T]$ and $b_i = \deg P_i$ are independent of ℓ.

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CRYSTALLINE AND p-ADIC PROOFS

The p-adic side. Berthelot's crystalline cohomology $H^i_{\text{cris}}(X/W(\mathbb{F}_q))$ is a Weil cohomology for $\ell = p$; Katz–Messing (1974) deduced from Deligne's theorem that Frobenius on $H^i_{\text{cris}} \otimes \mathbb{Q}$ has the *same* polynomial $P_i(T)$. Kedlaya (2006) gave a purely p-adic proof of (3) via rigid cohomology ("p-adic Weil II").

Compatible systems and lisse sheaves

DEFINITION (SERRE) — COMPATIBLE SYSTEMS OF GALOIS REPRESENTATIONS

K a global field, E a number field, S a finite set of places of K. A strictly compatible E-rational system is a family of continuous representations $\rho_\lambda : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(E_\lambda)$, one for each finite place λ of E, each unramified outside $S \cup \{v \mid \lambda\}$, such that for every place $v \notin S$ with $v \nmid \lambda$

$$\det\bigl(1 - \rho_\lambda(\mathrm{Frob}_v)\,T\bigr) \;\in\; E[T] \qquad \text{is independent of } \lambda.$$

Examples: $\{\mathbb{Q}_\ell(1)\}_\ell$, $\{V_\ell E\}_\ell$, and $\{H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)\}_\ell$ for X smooth projective (Deligne).

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X a connected normal scheme of finite type over $\mathbb{F}_p$, $\bar{x}$ a geometric point, $\pi_1(X, \bar{x})$ its étale fundamental group: {finite étale covers of X} $\simeq$ {finite sets with continuous $\pi_1(X, \bar{x})$-action} (SGA 1). A lisse $\overline{\mathbb{Q}}_\ell$-sheaf ($\ell$-adic local system) $\mathcal{F}$ of rank r on X is the same as a continuous representation

$$\rho_{\mathcal{F}} :\; \pi_1(X, \bar{x}) \longrightarrow \mathrm{GL}(\mathcal{F}_{\bar{x}}) \cong \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell).$$

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THE ANALOGY

Topology	Étale, over $\mathbb{F}_p$
manifold M, base point m	X normal over $\mathbb{F}_p$, geometric point $\bar{x}$
π_1^{top}(M, m)	$\pi_1(X, \bar{x})$: automorphisms of the universal pro-cover
covering spaces	finite étale covers
local systems = representations of π_1^{top}	lisse sheaves = continuous representations of π_1
loops	Frobenius classes F_x ($x \in |X|$) — dense (Čebotarev)

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DEFINITION — E-COMPATIBLE SYSTEMS OF LISSE SHEAVES

An E-compatible system of lisse sheaves on X is a family $\{\mathcal{F}_\lambda\}$, λ over the finite places of E not above p, $\mathcal{F}_\lambda$ a lisse E_λ-sheaf, such that $\det(1 - F_x t \mid \mathcal{F}_\lambda) \in E[t]$ is independent of λ for every $x \in |X|$. Example: $\{R^i f_* \mathbb{Q}_\ell\}_{\ell \neq p}$ for $f : Y \to X$ smooth projective.

THE ANALOGY

Topology	Étale, over $\mathbb{F}_p$
manifold M, base point m	X normal over $\mathbb{F}_p$, geometric point $\bar{x}$
$\pi_1^{\mathrm{top}}(M, m)$	$\pi_1(X, \bar{x})$: automorphisms of the universal pro-cover
covering spaces	finite étale covers
local systems = representations of π_1^{top}	lisse sheaves = continuous representations of π_1
loops	Frobenius classes F_x ($x \in |X|$) — dense (Čebotarev)

Deligne's conjecture (Weil II, Conjecture 1.2.10)

CONJECTURE (DELIGNE 1980; FORMULATION AFTER DRINFELD)

Let X be a normal connected scheme of finite type over $\mathbb{F}_p$, $\ell \neq p$, and $\mathcal{F}$ an *irreducible* lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X whose determinant has finite order. Then:

(i) $\mathcal{F}$ is *pure of weight* 0: every eigenvalue α of F_x is algebraic with $|\iota(\alpha)| = 1$ for all $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$;

(ii) there is a number field $E \subset \overline{\mathbb{Q}}_\ell$ with $\det(1 - F_x t \mid \mathcal{F}) \in E[t]$ for all $x \in |X|$;

(iii) for every finite place $\lambda \nmid p$ of E the eigenvalues of F_x are λ-adic units (they and their inverses are integral over $\mathbb{Z}[1/p]$);

(iv) *companions:* for every finite place $\lambda \nmid p$ of E (after possibly enlarging E) there is a lisse E_λ-sheaf $\mathcal{F}_\lambda$ on X with $\det(1 - F_x t \mid \mathcal{F}_\lambda) = \det(1 - F_x t \mid \mathcal{F})$ for all $x \in |X|$;

(v) the same for $\lambda \mid p$, with $\mathcal{F}_\lambda$ an overconvergent F-isocrystal (*crystalline companion*).

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THE MOTIVIC QUESTION BEHIND IT

Question (Drinfeld). Is such an $\mathcal{F}$, up to a Tate twist, a subquotient of $H^i(Y_{\bar{K}}, \overline{\mathbb{Q}}_\ell)$ for some variety Y over $K = \mathbb{F}_p(X)$? *Yes* for curves (Lafforgue); *open* for $\dim X > 1$.

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STATUS

Part	Curves	General X
(i) purity	Lafforgue 2002	Lafforgue (Bertini), Deligne
(ii) number field E	Lafforgue 2002	Deligne 2012; Esnault–Kerz
(iii) λ-adic units	Lafforgue 2002	Lafforgue, Deligne
(iv) ℓ'-adic companions	Lafforgue 2002	Drinfeld 2012 for X smooth (+ Chin); normal X open
(v) crystalline companions	Abe 2018	Abe–Esnault 2019, Kedlaya 2022 (X smooth)

Deligne's original motivation: (i)–(iv) imply that every lisse sheaf on X is mixed. On a curve, normal = smooth, so 'curves' means smooth curves.

The Langlands correspondence for $\mathrm{GL_r}$ (L. Lafforgue)

SETTING

X a smooth projective geometrically connected curve over $\mathbb{F}_q$, $F = \mathbb{F}_q(X)$ its function field, $\mathbb{A}$ its adèles, $|X|$ its closed points; fix $\ell \neq p$ and an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.

- $\{\sigma\}_r$: isomorphism classes of irreducible r-dimensional ℓ-adic representations of $\mathrm{Gal}(\bar{F}/F)$ that are lisse on some open $U \subset X$ and have determinant of finite order.
- $\{\pi\}_r$: cuspidal automorphic representations of $\mathrm{GL}_r(\mathbb{A})$ (their central characters automatically have finite order).

At $x \in |X|$ where they are unramified: $L_x(\sigma, T) = \det(1 - F_x T \,|\, \sigma)^{-1}$ and $L_x(\pi, T) = \prod_{i=1}^{r}(1 - z_i(\pi_x)T)^{-1}$, $z_i(\pi_x)$ the Hecke (Satake) parameters.

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THEOREM (DRINFELD r = 2; LAFFORGUE r ≥ 3, 2002)

There is a unique bijection $\{\pi\}_r \xrightarrow{\sim} \{\sigma\}_r$, $\pi \mapsto \sigma_\pi$, such that at every $x \in |X|$ where both are unramified

$$L_x(\pi, T) = L_x(\sigma_\pi, T), \qquad \text{i.e.} \quad \{\text{Hecke parameters of } \pi_x\} = \{\text{eigenvalues of } F_x \text{ on } \sigma_\pi\}.$$

It is compatible with the local correspondence at *all* places (Laumon–Rapoport–Stuhler). Proof: σ_π is realised in the ℓ-adic cohomology of the moduli stacks of rank-r Drinfeld shtukas, compared with the Arthur–Selberg trace formula.

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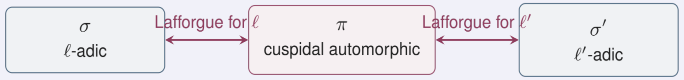
COMPANIONS BY TRANSPORT THROUGH π



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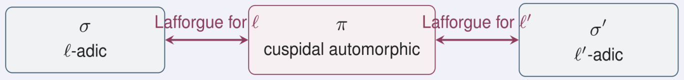

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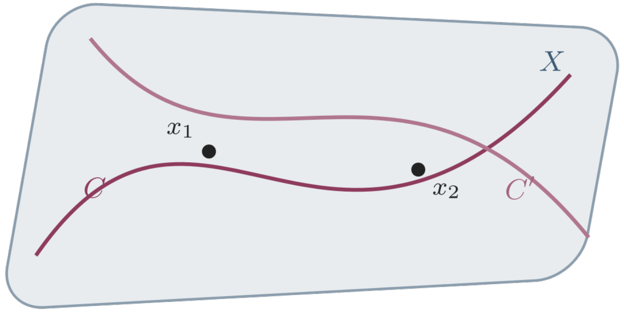
COROLLARY (LAFFORGUE, THM. VII.6): DELIGNE'S CONJECTURE FOR CURVES

For $\sigma \in \{\sigma\}_r$, lisse on $U \subset X$: (a) there is a number field E with $\det(1 - F_x T \,|\, \sigma) \in E[T]$ for all $x \in |U|$; (b) the eigenvalues of F_x are algebraic, of complex absolute value 1 under every embedding, and λ-adic units for all $\lambda \nmid q$; (c) for every prime $\ell' \neq p$ there is an irreducible ℓ'-adic σ' on U with the same Frobenius polynomials.
Why. (a) Hecke eigenvalues of cusp forms of a fixed level lie in a number field (finite-dimensional spaces with a $\mathbb{Q}$-structure). (b) σ_π lives in the pure cohomology of shtukas; on the automorphic side this is Ramanujan–Petersson. (c) Transport: $\sigma \mapsto \pi_\sigma \mapsto \sigma'$ via the correspondence for ℓ': the automorphic side does not know ℓ.

Reducing to curves: Bertini, finiteness, Wiesend's method

(i), (iii): SPACE-FILLING CURVES

Purity and λ-integrality are conditions at each closed point x. Space-filling/Bertini curves (Katz 1999–2001, Poonen 2004): for X smooth, $\mathcal{F}$ lisse and a finite $S \subset |X|$ there is a smooth curve $\varphi : C \to X$ through S such that pull-back $\langle \mathcal{F} \rangle \to \langle \varphi^* \mathcal{F} \rangle$ is an equivalence of Tannakian categories (same monodromy group); in particular $\varphi^* \mathcal{F}$ is irreducible with finite-order determinant. Now apply Lafforgue on C. [Lafforgue, Prop. VII.7; Deligne, *Finitude*, §1.5–1.9]



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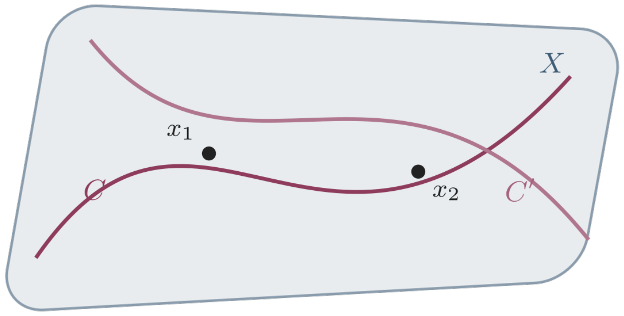
Deligne (2012), Esnault–Kerz: on a curve, up to twist by characters of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, there are only *finitely many* irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaves of given rank and bounded ramification (Lafforgue + Harder's finiteness for cusp forms). Feeding this uniform finiteness into a Bertini/Hilbert-irreducibility argument over the curves of X shows that the Frobenius traces at *all* closed points generate a number field.



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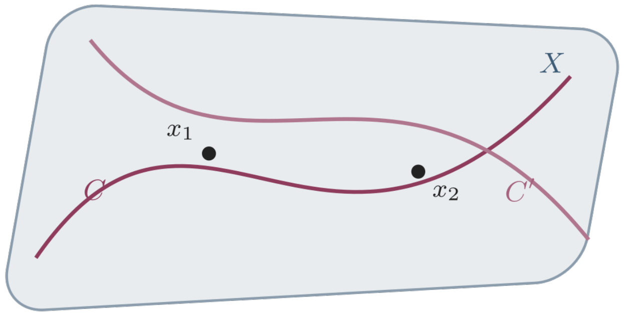

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(iv): THEOREM (DRINFELD 2012, AFTER WIESEND) — GLUING LOCAL SYSTEMS FROM CURVES

X regular of finite type over $\mathbb{Z}[1/\ell]$, $E_\lambda/\mathbb{Q}_\ell$ finite with ring of integers O. A rule $f : |X| \to \{1 + c_1 t + \cdots + c_r t^r : \ c_i \in O, \ c_r \in O^\times\}$ comes from a lisse E_λ-sheaf on X (unique up to semisimplification) if and only if

(a) for every regular curve $\varphi : C \to X$, the rule $\varphi^* f$ comes from a lisse E_λ-sheaf on C;

(b) there is a dominant étale $X' \to X$ such that on curves in X' these sheaves are tame.

Tools: Hilbert irreducibility (curves through prescribed points with $\pi_1(C) \twoheadrightarrow \pi_1(X)/H$ for almost-pro-ℓ quotients), Zariski–Nagata purity, Čebotarev, compactness of $\mathrm{Hom}(\pi_1(X)/H, \mathrm{GL}_r(O))$. For f = Frobenius polynomials of an ℓ'-adic $\mathcal{F}$: (a) is Lafforgue on curves; (b) holds since companions on curves are simultaneously tame (Deligne 1973). *Regularity is essential*: there are normal counterexamples.

Independence of ℓ for monodromy groups (Chin)

SETTING — ARITHMETIC MONODROMY GROUPS

X a smooth curve over $\mathbb{F}_q$, E a number field, $\mathbf{L} = \{\mathcal{L}_\lambda\}_{\lambda \nmid p}$ an E-compatible system on X which is semisimple and pure of some weight $w \in \mathbb{Z}$; fix a geometric point $\bar{\eta}$. The arithmetic monodromy group

$$G_\lambda := \overline{\rho_{\mathcal{L}_\lambda}\big(\pi_1(X, \bar{\eta})\big)}^{\mathrm{Zar}} \subset \mathrm{GL}(\mathcal{L}_{\lambda,\bar{\eta}})$$

is a (possibly disconnected) reductive group over E_λ, with identity component G_λ°.

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The kernel of $\pi_1(X, \bar{\eta}) \to G_\lambda \to G_\lambda / G_\lambda^\circ$ is the same open subgroup for every $\lambda \nmid p$; hence the finite groups $\pi_0(G_\lambda)$ are independent of λ.

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After replacing E by a finite extension F, there is a connected split reductive group G_0 over F such that for every finite place $\lambda \nmid p$ of F

$$G_\lambda^\circ \otimes_{E_\lambda} F_\lambda \;\cong\; G_0 \otimes_F F_\lambda.$$

The isomorphism can be chosen to identify a common split maximal torus T_0 (a *Frobenius torus*) and to match the tautological representations; it is then unique up to conjugation by $T_0(F_\lambda)$. [Generalised to smooth varieties of any dimension and to crystalline companions by D'Addezio (2020).]

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MOTIVIC MEANING

Heuristic: if $\mathbf{L}$ is "motivic", G_λ should be the λ-adic realisation of the motivic Galois group of the underlying motive (Serre; Larsen–Pink). Chin's theorem is the unconditional shadow of this: the group is determined, up to isomorphism, by data that do not depend on λ.

Sketch of Chin's proof: Frobenius tori and Grothendieck semirings

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STEP 1

Connect. By Serre's theorem the open subgroup $\ker(\pi_1 \to \pi_0(G_\lambda))$ is independent of λ; pulling $\mathbf{L}$ back to the corresponding finite étale cover $X' \to X$ replaces every G_λ by G_λ°. So assume all G_λ connected.

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Frobenius tori (Serre). For infinitely many $x \in |X|$ the Zariski closure of $\langle \rho_\lambda(F_x) \rangle$ is a maximal torus of G_λ, and it is the base change of an E-torus $A(x)$ determined by $\det(1 - F_x T)$ alone — *the same for all* λ. Inputs: E-rationality, purity and λ-adic integrality (Lafforgue), bounded p-adic valuations and denominators (Lafforgue, Chin). Enlarge E to split $A(x)$: every G_λ is split with the common maximal torus T_0.

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Sketch of Chin's proof: Frobenius tori and Grothendieck semirings

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STEP 1

Connect. By Serre's theorem the open subgroup $\ker\big(\pi_1 \to \pi_0(G_\lambda)\big)$ is independent of λ; pulling $\mathbf{L}$ back to the corresponding finite étale cover $X' \to X$ replaces every G_λ by G_λ°. So assume all G_λ connected.

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STEP 2

Frobenius tori (Serre). For infinitely many $x \in |X|$ the Zariski closure of $\langle \rho_\lambda(F_x) \rangle$ is a maximal torus of G_λ, and it is the base change of an E-torus $A(x)$ determined by $\det(1 - F_x T)$ alone — *the same for all* λ. Inputs: E-rationality, purity and λ-adic integrality (Lafforgue), bounded p-adic valuations and denominators (Lafforgue, Chin). Enlarge E to split $A(x)$: every G_λ is split with the common maximal torus T_0.

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Reconstruction. A connected split reductive group is determined up to isomorphism by $\big(T_0,\ \mathrm{Irr}(G) \subset K(G),\ \mathrm{ch}\big)$ [Chin 2003]; in fact the semiring $K^+(G)$ alone determines a connected reductive G [Kazhdan–Larsen–Varshavsky 2014]. Hence $G_\lambda \cong G_0 \otimes F_\lambda$ for all λ, with G_0 the split form over F of any one G_μ.

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UPSHOT

The λ-independent Frobenius data determine the Grothendieck semiring of the monodromy group (via companions), and the semiring determines the connected group (via reconstruction). Everything ultimately rests on Lafforgue's correspondence.

Drinfeld: the pro-semisimple completion of $\pi_1(X)$

DEFINITION — THE λ-ADIC PRO-SEMISIMPLE COMPLETION

X a smooth connected variety over $\mathbb{F}_p$, $\Pi := \pi_1(X, \bar{x})$, λ a place of $\overline{\mathbb{Q}}$ above $\ell \neq p$, $\overline{\mathbb{Q}}_\lambda$ the completion. The λ-adic pro-semisimple completion of Π is

$$\widehat{\Pi}_\lambda := \varprojlim_{(G,\rho)} G, \qquad (G, \rho): \; G \text{ a semisimple algebraic group over } \overline{\mathbb{Q}}_\lambda \text{ (not necessarily connected)}, \;\; \rho : \Pi \to G(\overline{\mathbb{Q}}_\lambda) \text{ continuous with Zariski-dense image.}$$

It is a pro-semisimple group scheme with $1 \to \widehat{\Pi}_\lambda^\circ \to \widehat{\Pi}_\lambda \to \Pi \to 1$ and a canonical $\Pi \to \widehat{\Pi}_\lambda(\overline{\mathbb{Q}}_\lambda)$; its identity component is simply connected. Tannakian description: $\mathrm{Rep}(\widehat{\Pi}_\lambda) =$ semisimple λ-adic local systems on X all of whose irreducible constituents have determinant of finite order.

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THE COARSE GROUPOID AND THE FROBENIUS DATA

For E algebraically closed let $\mathrm{Pro\text{-}ss}(E)$ be the groupoid whose objects are pro-semisimple groups over E and whose morphisms $G_1 \to G_2$ are isomorphisms of group schemes *modulo conjugation by* G_2°. Base change $\mathrm{Pro\text{-}ss}(\overline{\mathbb{Q}}) \to \mathrm{Pro\text{-}ss}(\overline{\mathbb{Q}}_\lambda)$ is an equivalence, so $\widehat{\Pi}_\lambda$ defines an object $\widehat{\Pi}_{(\lambda)} \in \mathrm{Pro\text{-}ss}(\overline{\mathbb{Q}})$, equipped with Frobenius data

$$\Pi_{\mathrm{Fr}} := \{F_{\tilde{x}}^n\} \longrightarrow [\widehat{\Pi}_{(\lambda)}](\overline{\mathbb{Q}}) \twoheadrightarrow \Pi, \qquad [G] := \text{quotient of } G \text{ by } G^\circ\text{-conjugation},$$

which take values in $\overline{\mathbb{Q}}$ (not just $\overline{\mathbb{Q}}_\lambda$) by Lafforgue, Prop. VII.7.

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THEOREM (DRINFELD 2018)

Let X be smooth and λ, λ' places of $\overline{\mathbb{Q}}$ not dividing p. There is a *unique* isomorphism $\widehat{\Pi}_{(\lambda)} \xrightarrow{\sim} \widehat{\Pi}_{(\lambda')}$ in $\mathrm{Pro\text{-}ss}(\overline{\mathbb{Q}})$ compatible with the Frobenius data. Hence $\widehat{\Pi} := \widehat{\Pi}_{(\lambda)} \in \mathrm{Pro\text{-}ss}(\overline{\mathbb{Q}})$ is well defined and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant, with $\widehat{\Pi}/\widehat{\Pi}^\circ = \Pi$. Chin's theorem is the statement for $\widehat{\Pi}^\circ$. For $\dim X = 1$ the same holds for $\lambda \mid p$ (overconvergent F-isocrystals; Abe 2018).

Why $\hat{\Pi}$ exists, and what it should be

IDEA OF THE PROOF

$K^+(\widehat{\Pi}_{(\lambda)})$:= Grothendieck semiring of $\mathrm{Rep}(\widehat{\Pi}_{(\lambda)})$ with the exterior-power operations (a λ-*semiring*): a "poor man's" Tannakian category. Restricting characters to Frobenius elements embeds it into the functions $\Pi_{\mathrm{Fr}} \to \overline{\mathbb{Q}}$, and by Lafforgue/Drinfeld (companions) the image does not depend on λ; likewise for every open $U \subset \Pi$. Reconstruction (after Kazhdan–Larsen–Varshavsky): a pro-semisimple G with $\pi_0(G) = \Pi$ is determined by the λ-semirings $K^+(G \times_\Pi U)$, U open, provided $H^2(U, \mathbb{Q}/\mathbb{Z}) = 0$ — true for affine curves; general X by reduction to curves via Hilbert irreducibility. Plain semirings fail for disconnected G (e.g. $(\mathbb{Z}/2) \ltimes \mathrm{SL}_{2n+1}$): the λ-operations are essential.

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MOTIVIC INTUITION

Let $K = \mathbb{F}_p(X)$. Representations of Π are Galois representations of K that are *unramified along* X. Conjecturally $\widehat{\Pi}$ (or its pro-reductive variant $\widehat{\Pi}^{\mathrm{mot}}$) is the λ-adic realisation of the quotient of the motivic Galois group $G_{\mathrm{mot}}(K)$ classifying motives over K with good reduction along X — "motivic local systems on X". Drinfeld: $\widehat{\Pi}$ should come from a pro-semisimple gerbe over $\mathbb{Q}$ mapping to $B\Pi$. Unconditionally we only have $\widehat{\Pi} \in \text{Pro-ss}(\overline{\mathbb{Q}})$; for $\dim X > 1$ it is unknown whether every weakly motivic local system on X comes from a motive over K.

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THE ℓ-INDEPENDENCE LADDER

Frobenius polynomials
Weil I (Deligne 1974)

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Questions?