# A Review of Three-Dimensional Geometry and Topology

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## 0 Introduction

There are many techniques to study the rich theory of 2-manifolds. The geometry splits into three cases, according to their curvature; spherical geometry (positive curvature), Euclidean geometry (flat) and hyperbolic geometry (negative curvature). There is also a sweet topological characterisation; every closed 2-manifold is topologically determined by it's Euler characteristic and orientability. One can also study a lot of this theory by putting on a complex structure, where there are essentially the same classifications of Riemann surfaces, as quotients of the (only) simply connected ones - the Riemann sphere, the complex plane or the open disk. The theory of functions is best understood algebraically, and a more general theory can be developed in algebraic geometry. The aim of this essay is to understand a bit of the theory of 3-manifolds, where things get a little weirder. The lack of a complex analogue means our best tools are topological and geometric, some of which are adapted from the 2 dimensional case, the hope is to understand some geometry of 3-manifolds. This essay will be split into two sections, the first will be on 2-manifolds and will largely be quite structured with definitions, theorems etc. but the second half, on 3-manifolds will be a more talkative, generally the bits that are new in any sense will be like this with pictures (even in section 1). The guide is [1] *Three-Dimensional Geometry and Topology*, W.P.Thurston.

## 1 Manifolds

### 1.1 Basics

A *manifold* is a nice topological space that locally looks like Euclidean space - but this is all, how we glue these bits drastically changes its topology. It is like a zero knowledge proof, we simply convey to the other party that we are in possession of this object without revealing anything about it. The global point properties ensure nothing too weird is going on, such as a double origin or points being hard to distinguish (by open sets, say). The localness means that often we can work with a linear approximation of it, namely its tangent plane, where we have access to the theory of linear algebra. There are two neat ways to define such an object:

**Definition:** A *k-manifold* $X$ is a second-countable, Hausdorff topological space, such that $\forall\, x \in X$, $\exists$ a neighbourhood $U$ of $x$ such that $U$ is diffeomorphic to an open set $V$ in $\mathbb{R}^k$.

The diffeomorphism $\phi : V \to U$ is called a *local parameterisation* of $U$, $\phi^{-1}$ is a *change of coordinates* and the pair $(\phi^{-1}, U)$ is called a *chart*, although we sometimes drop $U$. An *atlas* is a collection of charts that cover the space. For charts $\phi, \psi$ say $\psi \circ \phi^{-1}$ is a *transition map* where appropriately defined.

Using charts, we can make a slightly more abstract definition, imposing a compatibility condition on charts:

**Definition:** A *smooth k-manifold* $X$ is a second-countable, Hausdorff topological space endowed with a smooth atlas i.e the transition maps are smooth

The condition of smoothness of transition maps can be changed, and it is often more useful to consider a general definition. The idea is that we make manifolds by gluing pieces of Euclidean space, but we can put restrictions on how we glue them to give our manifold more structure - this translates to putting conditions on the set of transition maps, these should satisfy some natural properties:

**Definition:** A *pseudogroup* on a topological space $X$ is a set $\mathcal{G}$ of homeomorphisms of $X$ satisfying:

(i) the domains of elements $g \in \mathcal{G}$ cover X

(ii) the restriction of $g \in \mathcal{G}$ to any open set in the domain of $g$ is also in $\mathcal{G}$

(iii) the composition $g_1 \circ g_2$, where defined, is in $\mathcal{G}$

(iv) the inverse of any element in $\mathcal{G}$ is in $\mathcal{G}$

(v) if $g : U \to V$ is a homeomorphism of open sets in $X$ and $U$ is covered by open sets $U_\alpha$ such that $g|_\alpha$ is in $\mathcal{G}$ then $g$ is in $\mathcal{G}$

So we can now model the gluings on a pseudogroup $\mathcal{G}$:

**Definition:** Let $\mathcal{G}$ be a pseudogroup on $\mathbb{R}^n$. An n-dimensional $\mathcal{G}$-*manifold* is a Hausdorff, second countable topological space $X$ with a $\mathcal{G}$-atlas, i.e the transition maps are in $\mathcal{G}$.
If $\mathcal{G}$ is the pseudogroup of all homeomorphisms of $\mathbb{R}^n$ then $X$ is a *topological manifold.* If $\mathcal{G} = \mathcal{C}^r$, the set of all continuous functions with continuous partial derivatives of total order $r$ then $X$ is a *differentiable manifold.* If $\mathcal{G} = \mathcal{C}^\infty$ then $X$ is a *smooth manifold.* In the case $n = 2$ we may identify $\mathbb{R}^2$ with $\mathbb{C}$ and a homeomorphism on $\mathbb{R}^2$ is holomorphic if and only if it is conformal and orientation preserving - in this case we say $X$ is a *Riemann surface.* Interestingly,

second countability of Riemann surfaces comes for free - this is called *Radó's theorem* [9].

In some sense, we really only need to study smooth manifolds and Riemann surfaces (for the two dimensional case). Indeed Whitney [6] showed any differentiable manifold can be given a unique smooth structure. Also any orientable surface with a Riemannian metric can be given a complex structure, using *isothermal coordinates* - see [7] for more details. However, one doesn't get a good theory of topological manifolds, at least in higher dimensions - there is no real complex analogue other than the non-commutative quaternions (these are in dimension 4, in dimension 8 we have octonians but these are not associative and is the last of the *division algebras*) and in dimensions greater than 3 we can smoothen some topological manifolds in several inequivalent ways - see *exotic spheres*, Milnor [8]. From now we only look at compact smooth manifolds or Riemann surfaces.

Another benefit of using a more abstract definition is that we can easily define any local things like continuity or smoothness of maps between manifolds through the charts, i.e for $f : X \to Y$ we define properties through $\psi \circ f \circ \phi^{-1}$

It should be noted, apriori $k$ is not well-defined. One way to properly define the dimension of a manifold is by its linearisation.

Recall that a for a map $g : V \to \mathbb{R}^n$ where $0 \in U \subset \mathbb{R}^m$, has derivative $d\phi_0 : \mathbb{R}^m \to \mathbb{R}^n$, a linear map.

**Definition:** For $x \in X$ the *tangent plane of $X$ at $x$* is $T_xX = d\phi_0(\mathbb{R}^k)$ where $\phi$ is any chart at $x$ and $\phi(0) = x$

**Proposition:** $T_xX$ is independent of a local parameterisation and $\dim T_xX = \dim X$

The proof is pretty straightforward, writing charts $\phi, \psi$ as $\phi = \psi \circ h$ where $h = \psi^{-1} \circ \phi$. By uniqueness of dimensions of vector spaces we now have that $\dim X$ is well defined. Access to linear algebra by approximating $X$ at $x$ as $T_xX$ gives us some nice results, such as for $f : X \to Y$ a smooth map of smooth manifolds have $f^{-1}(y)$ is a submanifold of $X$ with $\operatorname{codim} f^{-1}(y) = \dim Y$ by rank-nullity on tangent planes, assuming $y$ is a *regular value*; the points $f(x)$ where $df_x$ is surjective. In fact most values are regular (they form a dense set, known as *Sard's theorem*). A nice application of this is to show the orthogonal group $O(n)$ is a manifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$ by observing $O(n) = f^{-1}(I)$ where $f : M(n) \to S(n)$ is the map from the set of $n \times n$ matrices to the set of symmetric $n \times n$ matrices; one only needs to see that $I$ is a regular value and these two spaces are vector spaces of dimension $n^2$ and $\frac{n(n+1)}{2}$ respectively.

This hopefully gives a taste into some basic results in differential geometry, but we will focus more on construction of 2 and 3 dimensional manifolds, their isometries and various geometric properties, as there are lots of lovely results in low dimensions.

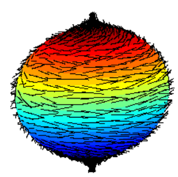
## 1.2 Topology of 2-Manifolds

First we look at a topological classification of 2-manifolds, remarkably this can be done with just two pieces of data - the Euler characteristic, and orientability. The first definition is combinatorial, but is an important invariant of a surface.

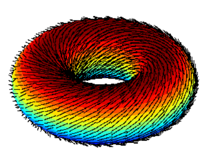
**Definition:** For a topological surface $X$ admitting a cell decomposition $C$, define its *Euler Characteristic* to be $\chi(X) := \chi(C) = \sum_i (-1)^i C_i$ where $C_i$ is the number of $i$-dimensional faces in $C$.

So this number is really associated to, in some sense, a very simple approximation of a surface - it is quite cool that we only need to work with a bunch of triangles glued together that looks a bit like the surface to almost classify surfaces topologically. There are several ways to arrive at the result that this number is independent of the cell decomposition, they are conceptually easy to understand but require filthy rigour to make logical sense of, these proofs are not enlightening so only a sketch of the ideas will be given.

**Sketch 1:** Put charges of +1 on the even dimensional faces, and -1 on the odd dimensional faces, we then displace these charges using a vector field, where most charges cancel except at the zeroes. The total sum of charges is the Euler characteristic and after displacing this only depends on the vector field, so we are done. As an example we can apply the vector field that rotates the sphere, the charge only accumulates at the poles of rotation, quantitatively this gives a total charge of +2, which is the desired Euler number - this is an example of the "Hairy Ball theorem" [2].



(a) The two bits we can't comb add up to the Euler characteristic



(b) We can comb a torus!

More formally, consider a vector field $v : X \to \mathbb{R}^k$. With the above brief de-

scription, have for $x \in X$ with $v(x) = 0$, the *index* of $v$ at $x$ is the degree of the map $\frac{v}{||v||}$ of a punctured neighbourhood of $x$ to $S^{k-1}$, denoted $I(v, x)$.
If $dv_x$ is non-singular, then the index of x is either 1 or -1 depending on the sign of determinant of the derivative. This comes from the fact that a diffeomorphism $f$ with $f(0) = 0$ is homotopic to its derivative via $H(t, x) = \frac{f(tx)}{t}$ for $t \neq 0$ and $H(0, x) = df_0(x)$, hence by the partition into connected components of $GL(k, \mathbb{R})$ by the sign of determinant, we get the result. To finish one proves the following:

**Theorem** *(Poincare-Hopf):* For a vector field with isolated zeroes on a compact manifold, the sum of its indices is the Euler characteristic

The hard bit of this result is showing the existence of such a vector field. Assuming this, one can show that this index sum is in fact the degree of the *Gauss map*, so the sum is independent of the vector field. Computing this for one such finishes the job. Details can be ironed out and a satisfying presentation of this argument is given in [3].

**Sketch 2:** This one uses homology theory. To define the homology groups we try that to detect a n-dimensional hole one can find a n-dimensional loop. Such a loop will show the existence of a hole so long as it is not the boundary of an (n+1)-dimensional face. Formally one sets up a chain complex, vector spaces spanned by the faces of a given dimension, and then finds a suitable boundary function which sends faces to their oriented boundary. Then the quotient group of cycles (things that add to 0) by the boundaries (kernels of boundary maps) will give the various (dimensional) homology groups. The hard bit is showing these are homotopy invariant, and it can get a little messy with tons of definitions, but one does meet this in an undergraduate algebraic topology course.

The next bit of data to look at is orientability, where if one holds a compass and takes any walk, the compass should look oriented in the same away upon returning; it should not have flipped (say) its clockwise direction. For a surface embedded in Euclidean space we can define orientability by looking for a consistent choice of a normal vector to the surface:

**Definition:** Say a smooth surface $X$ is *two-sided* or *orientable* if it admits a continuous global choice of a unit normal vector. Formally, there is a smooth map $N : X \to S^2$ such that $\forall x \in X$, $N(x)$ is orthogonal to $T_xX$.
For example, a surface in $\mathbb{R}^3$ with local parameterisation given by $\sigma(u, v)$, have a local positive normal given by the vector product $\frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}$. By imposing differentiability of transition maps, we can have a slightly better definition:

**Definition:** Say a smooth manifold $X$ is *orientable* if it admits an atlas for which the transition maps are orientation preserving diffeomorphisms, that is for any $x \in X$ and any $\phi$, $\psi$ charts at $x$, have $\det(d(\phi \circ \psi^{-1})_x) > 0$

For a surface $X$, one can also detect orientability by algebraic means. Namely it is orientable if and only of its first homology group is torsion free. There will be a factor of $\mathbb{Z}/2\mathbb{Z}$ coming from the loop going down the center circle of the Möbius band for non-orientable surfaces. This comes from the characterisation of orientability of surfaces: it is not orientable if and only if it contains a sub-surface homeomorphic to the Möbius band. We have the following neat characterisation:

**Theorem:** A 2-manifold is uniquely determined (up to homeomorphism) by its Euler characteristic and its orientability.

A useful way to see this is by gluing the edges of 2n-gons. Consider a regular 2n-gon, its boundary is a graph and convention is to apply a direction on the edges in an anti-clockwise manner. Then identify pairs of edges and glue them by a homeomorphism of the interval (considering edges themselves as homeomorphic to an interval), there are only two isotopy classes of homeomorphism of the interval, given by whether they are increasing or decreasing; if we consider the interval to be $[-1, 1]$ then the any homeomorphism of $I$ to itself is *isotopic* to the identity or reflection about 0 (isotopic meaning that the they are homotopic and every slice of the homotopy - i.e $f_t(x) = H(x, t)$ for $t$ fixed - is itself a homeomorphism). The glued surface will be orientable if and only if all edges are paired with an orientation reversing homeomorphism - this can be seen by travelling along the loop going from the midpoint of one to the other inside the polygon.

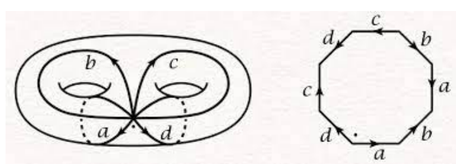



Figure 2: Gluing an octagon to get a genus two surface

By gluing the 4g-gon via the pattern $a_1 b_1 a_1^{-1} b_1^{-1}$... $a_g b_g a_g^{-1} b_g^{-1}$ as in figure 2, we obtain a a homeomorphism type of orientable surfaces. The number $g$ is the *genus* of the surface, and it is called the *g-holed torus.* Since for the boundary graph the number of vertices $V = 1$, the number of edges $E = 2g$ and the number of faces $F = 1$ we have that the Euler characteristic is $V - E + F = 2 - 2g$ By gluing the 2g-gon via the pattern $a_1 a_1$... $a_g a_g$ we get a homeomorphism type of non-orientable surfaces. The number $g$ is the *non-orientable genus* of the surface and the Euler characteristic is $2 - g$.

For example, on a digon there are two gluings. The orientation reversing homeomorphism gives the *2-sphere*, $S^2$ with $\chi(S^2) = 2$ and the orientation preserving homeomorphism gives the *real projective plane*, $\mathbb{RP}^2$ with $\chi(\mathbb{RP}^2) = 1$. Figure 3 also shows $\mathbb{RP}^2$ is the quotient of $S^2$ by the action of the antipodal map, which gives another way to compute its Euler characteristic since its an *orbifold* coming from a degree 2 cover.

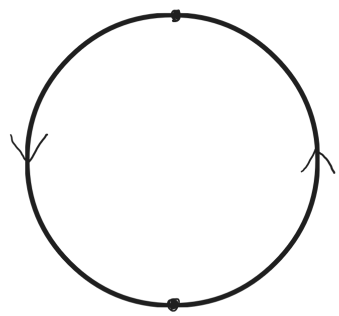


Figure 3: Digon with identification

By removing a disk from S we get a disk with attached strips, some of which are twisted, this can be seen by removing a small neighbourhood of each vertex in the polygon (and thus a small neighbourhood of the only vertex on our surface) and then pulling out the edges and gluing with an appropriate twist. By cutting the polygon up, and re-gluing we can ensure there is at most only ever one twist. This fact can be better seen by the classification:

**Definition:** The *connected sum* of two k-manifolds $X$, $Y$ is defined to be the surface $X\#Y$ obtained by removing a k-disk, $D^k$, from $X$ and a k-disk from $Y$ and gluing the boundary spheres.

**Claim:** For surfaces $X$ and $Y$ have $\chi(X\#Y) = \chi(X) + \chi(Y) - 2$
Indeed, choose triangulations for each, and glue two triangles together, then the 3 vertices of the triangle from $X$ are glued to 3 vertices from the triangle from $Y$, same with edges so $V - E$ is unchanged. The number of faces decreases by 2 as we lose one from each triangle.

We can now argue the above twisting using this notion: for any surface first compute its Euler characteristic $\chi$, and its orientability, denote this pair by $(\chi, +)$ if it is orientable and $(\chi, -)$ if it is non-orientable. Now we observe, by uniqueness, it is (homeomorphic to) one of:

(i) $(2, +)$: The 2-sphere, $S^2$

(ii) $(-2m, +)$: The connected sum of $m + 1$ tori, $T^2\#...\#T^2$

(iii) $(-2m,-)$: The connected sum of $m$ tori and a Klein bottle, $T^2\#...\#T^2\#K$

(iv) $(-2m+1,-)$: The connected sum of $m$ tori and a real projective plane, $T^2\#...\#T^2\#\mathbb{RP}^2$

Of course from cutting and gluing the polygons or by the classification we also have $K = \mathbb{RP}^2\#\mathbb{RP}^2$. From above we also see that the set of closed surfaces with operation # is a commutative *monoid*, with identity $S^2$. There are many other gluing patterns we can have on a 2n-gon.

**Claim:** The number of gluings of a 2n-gon to produce orientable surfaces is $(2n-1)!! = \frac{(2n)!}{2^n n!}$.
Indeed pick an edge, then there are $2n-1$ edges we can pair up with that one. Now pick another edge from the remaining $2n-2$, there are $2n-3$ choices for its pair and so on, so we get a total of $(2n-1)\times(2n-3)\times...\times 1 = (2n-1)!!$. Alternatively, there are $\binom{2n}{2}$ choices for our first pair, then $\binom{2n-2}{2}$ for the next and so on. The order in which we glue these pairs is not important, so we get $\frac{1}{n!}\binom{2n}{2}\binom{2n-2}{2}...\binom{2}{2} = \frac{(2n)!}{2^n n!}$

**Claim:** The number of gluings of a 2n-gon is $\frac{(2n)!}{n!}$
Indeed, for each pair above, there are two possible ways to glue them, given by the orientation of the homeomorphism.
There are clearly lots of gluing patterns, this number is exponential but the number of topological types is very small - e.g. in the case of gluings producing orientable surfaces, by Euler's formula we have $V - E + F = V - n + 1 = 2 - 2g$, but $1 \leq V \leq n+1$, with upper equality if and only if the surface is $S^2$, or equivalently, the graph is a *tree* (has no cycles or loops), so there are many repeats.

There is a really nice way to compute the number orientable gluings of genus $g$ of a 2n-gon, let $\epsilon_g(n)$ be this number. One can prove inductively using the above equivalence of the graph being a tree and the surface being $S^2$ that $\epsilon_0(n) = Cat_n = \frac{1}{n+1}\binom{2n}{n}$, the $n^{th}$-*Catalan* number.
In fact much more can be said - consider the generating function $T_n(N) = \sum_{g=0}^{\infty} \epsilon_g(n) N^{n+1-2g} = \sum_\sigma N^{V(\sigma)}$ where the second sum is taken over all gluings $\sigma$ giving orientable surfaces and $V(\sigma)$ is the number of vertices of the graph on the surface produced by that gluing. Consider also the generating function of this generating function, $T(N,s) = 1 + 2Ns + 2s\sum_{n=1}^{\infty} \frac{T_n(N)}{(2n-1)!!} s^n$ then we have:

**Theorem ([4]):** $T(N,s) = (\frac{1+s}{1-s})^N$

This clearly holds for $N = 1$ but is hard for general $N$. Using this we can find a recurrence relation for our number:

**Corollary:** $(n+2)\epsilon_g(n+1) = (4n+2)\epsilon_g(n) + (4n^3-n)\epsilon_{g-1}(n-1)$ and

$$\epsilon_g(0) = \begin{cases} 1 & \text{if } g = 0 \\ 0 & \text{otherwise} \end{cases}$$

To prove the above theorem, Harer and Zagier showed a rather remarkable fact:

**Proposition:** $T_n(N) = \int_{\mathcal{H}_N} tr(H^{2n})d\mu(H)$ where $\mathcal{H}_N$ is the space of $N \times N$ *Hermitian* matrices and the measure is $d\mu(H) = (2\pi)^{\frac{-N^2}{2}} 2^{\frac{N^2-N}{2}} exp\{-\frac{1}{2}tr(H^2)dv(H)\}$ Note the ordinary volume form is $dv(H) = \prod_{i=1}^N dx_{ii} \prod_{i<j} dx_{ij}dy_{ij}$ given by the usual basis of Hermitian space viewed as a subspace of $\mathbb{R}^{N^2}$ (real $x_{ii}$ on the diagonal, complex $x_{ij} + iy_{ij}$ in upper triangle)

A similar integral over the space of *symmetric* matrices gives a generating function of all gluings of a 2n-gon. The key to this relationship is *Wick's formula*, which helps calculate the average of a product with respect to a Gaussian measure, using couplings. These couplings are exactly how we pair the edges to glue. A purely combinatorial proof is given by Lass [5].

The polygons also reveal other symmetries of surfaces. For example if we take the torus as a gluing of the square we can reflect along the diagonal to get a symmetry of order two. The loops $a$ and $b$ are switched, i.e we have switched the coordinate factors of $S^1 \times S^1 = T^2$. Can we realise this symmetry of an embedded torus in $\mathbb{R}^3$? If we consider the torus as a circle revolved around the $z$-axis as in figure 4, then the loop $b$ will always go around this, no matter what symmetry of $\mathbb{R}^3$ we apply, whereas the loop $a$ will not. This axis will stay in the unbounded component of the complement of the torus, so $a$ and $b$ can't be switched. Another way to say this is to look at the disks which these loops bound in each of these components and follow the same argument.

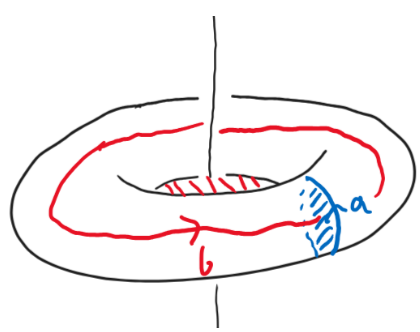



Figure 4: The loops $a$ and $b$ bounding disks of different components of $\mathbb{R}^3 - T$

However, if instead we put the torus in the 3-sphere, $S^3$ we can switch them.

This is easiest seen by considering $S^3$ as the *one-point compactification* of $\mathbb{R}^3$ (see *stereographic projection* in section 1.3). The idea is that lines in $\mathbb{R}^3$ are circles in $S^3$ passing through $\infty$, so we can move infinity around on the $S^1$'s to make them switch positions. Visually one can do this by moving the torus over $\infty$, it is quite fun to do.

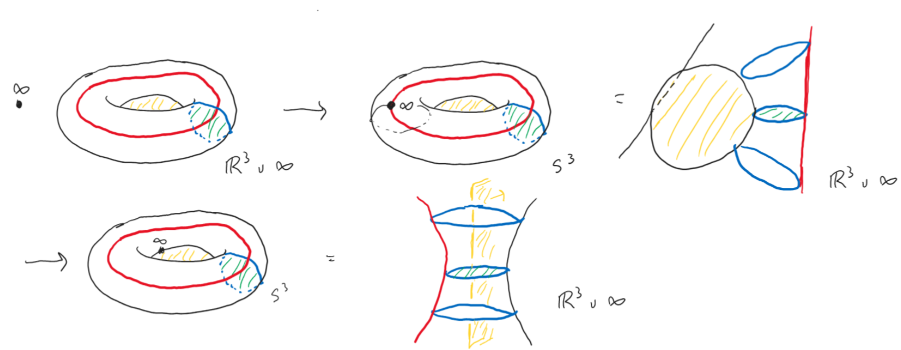



Figure 5: Moving a torus over infinity so it is now inside

Another way to see this symmetry take place in $S^3$ is to view the torus as the complement of the knots given by the $z$-axis (labelled A in figure 6) and a circle going around it (labelled B). Then the red loop $b$ goes around the $z$-axis and the loop $a$ goes around the circle, but in $S^3$ the $z$-axis is just a circle through $\infty$, so we can easily switch these two circles linked together and our loops $a$, $b$.

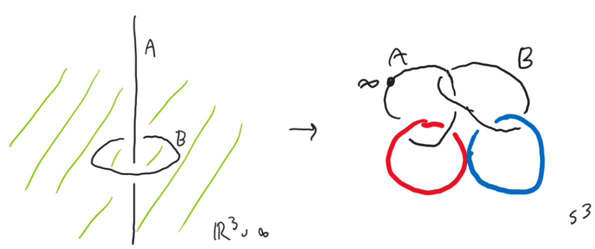



Figure 6: The green is homotopic to a torus generated by the red and blue loops

Drawing graphs on surfaces is a fun problem, when are no edge crossings (two edges can only meet at a vertex). If a graph can be drawn in the plane is such a way, we say it is *planar*, there is a nice theory of this. Given any drawn planar graph we can triangulate it, which reveals that $3F \leq 2E$ and so by Euler's formula we have $E \leq 3(V-2)$, so the number of edges is linear, out of a possible

$\binom{V}{2}$ graphs on $V$ vertices. Rather than triangulating, we can instead consider the *girth* of the graph, which is the length of the shortest cycle to get $gF \leq 2E$, and so $E \leq \frac{g}{g-2}(V-2)$. Such a bound tells us that a *complete* graph on 5 vertices, $K_5$ is not planar, and neither is the complete *bipartite* graph on classes each of size 3, $K_{3,3}$. In fact the converse is basically true:

**Theorem** (*Kuratowski*): A graph is planar if and only if it has no subdivision of a $K_5$ or a $K_{3,3}$

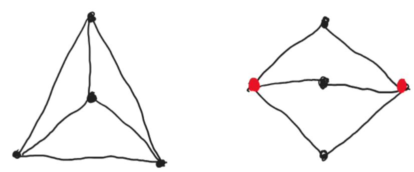


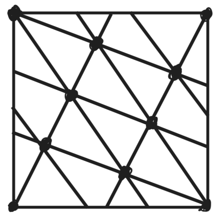
Figure 7: Plane drawings of $K_4$ and $K_{2,3}$

A *subdivision* of a graph just consists of adding vertices to edges. We can perform similar calculations on a surface with Euler characteristic $\chi$ to get $E \leq \frac{g}{g-2}(V-\chi)$. The *chromatic number* of a graph is the fewest number of colours we need to colour the vertices such that no two adjacent vertices have the same colour, for example the complete graph $K_n$ has a chromatic colour of $n$. Using this bound and considering a minimal graph of fixed chromatic colour it is easy to show:

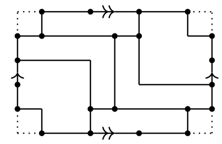
**Theorem** (*Heawood*): The chromatic number of a graph on a surface with Euler characteristic $\chi \leq 0$ is bounded above by $\frac{7+\sqrt{49-24\chi}}{2}$. In particular $K_n$ can be drawn on surface of genus $g$ only if $g \geq \frac{(n-3)(n-4)}{12}$

In fact the lower bound is sharp, but this is very hard to prove. Interestingly if one subs in $\chi = 2$ we get the 4-colour theorem, but this is not a valid proof! In the case $n = 7$ or equivalently (by sharpness) $\chi = 0$, it means we should be able to draw a $K_7$ on the torus. The *dual* of any graph is obtained by swapping the faces for vertices and vertices for faces: this is done by placing a vertex at each face, and drawing an edge between two vertices if the corresponding faces share an edge. By taking the dual of the $K_7$ on the torus, we get a map with 7 faces, each having to be colour differently (i.e each face is bordered with every other face).

There is an easier way to do this. To obtain the $K_7$ we took a square grid and rotated it and put a square on top of it with edges identified to glue to a torus. An easier way to obtain 7 faces all bordering each other is to tile the plane with hexagons, so any hexagon borders 6 others, and then draw a hexagon



(a) $K_7$ on a torus



(b) It's dual

on this tiling, with edges identified to form a torus. We can do this by observing that such a hexagon surrounded by 6 others has 18 vertices on the outside, so we start with one and connect it to the third in (say) an anticlockwise manner and repeat, since $\frac{18}{3} = 6$ we get our hexagon that we want and just have to glue its edges correctly and (tediously) check everything works out.

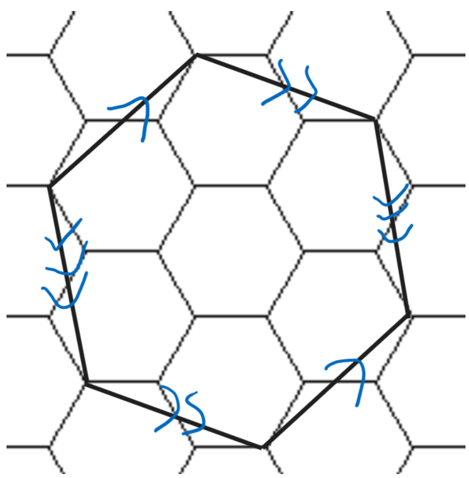


Figure 9: $K_7$ on hexagonal torus

In fact this hexagonal torus reveals an order 3 symmetry. If we draw this in $\mathbb{R}^3$ (figure 10) by considering the loop $ac^{-1}$ and the twisted loop $ba$ we see by a similar argument above that this symmetry cannot be realised in $\mathbb{R}^3$. We can try things for a higher genus gluing, for example a genus two surface has an obvious order 2 symmetry in $\mathbb{R}^3$. We can also find an order 3 symmetry in this space by gluing two *pairs of pants* in a different way (see section 1.3 for this definition). This shows we have the product of a dihedral group $D_6$ and reflection $C_2$ as a subgroup of the symmetries of the embedded genus two surface i.e. $C_2 \times D_6$.

We can also obtain a genus two surface from gluing an octagon or a decagon,

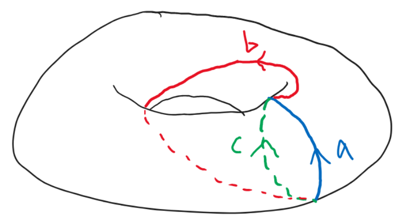



Figure 10: Twisted loop on hexagonal torus

which have order 8 and order 10 symmetries respectively (by rotating the polygon). To see these can't be realised in $\mathbb{R}^3$ or $S^3$ we see that the genus 2 surface bounds 6 disks, so any symmetry can be sent to a an element of $S_6$ the symmetric group acting on a set of size 6, and here there is no cycle of order 8 or 10. One can also argue this by looking at the finite subgroups of $SO(3)$ and the number of fixed or periodic points of the above rotations.

### 1.3 Geometry of 2-manifolds

We now look at the geometry of surfaces, studying lengths, angles, area and curvature. A surface will generally have varying (Gaussian) curvature, such as on a torus in $\mathbb{R}^3$ we can find directions that curve positively, look flat or curve negatively. However, we can find surfaces from each topological type that have a constant curvature, and it turns out this classifies the geometry of 2-manifolds via a lovely theorem that relates the overall curvature to the Euler characteristic, known as the *Gauss-Bonnet* theorem.

We will mostly study *geodesics*, these are length minimising paths. Formally one defines the *energy* of a curve $\gamma$ to be $E(\gamma) := \int_a^b ||\gamma'(t)||^2 dt$ and defines a geodesic to be a critical point of the energy functional $E(\gamma_s)$ for any family of *homotopic* paths to $\gamma$ with the same endpoints. Differentiating, we get the geodesic equations - this means geodesics are naturally locally length minimising. We also get that about any point there exists a geodesic in any direction, and that this is unique - this comes from the theorem of *Lindelöf - Picard*. The energy does depend on parameterisation, and using basic inequalities one sees that geodesics are naturally parameterised by arc length, that is $|\gamma'(t)| = 1$. We first recall some definitions.

**Definition:** The *first fundamental form* on a surface $S \subset \mathbb{R}^3$ at a point $p \in S$ is the quadratic form $I_p(w) = \langle w, w \rangle = |w|^2$ on $T_pS$. This defines a *Riemannian metric* on our surface.
If $S$ is locally parameterised by $\phi$ then writing $w \in T_pS$ as $\dot\alpha(0)$ where $\alpha$ is a curve on $S$ and $\alpha(0) = p$ we have $I_p(\dot\alpha(0)) = E(\dot u)^2 + 2F\dot u\dot v + G(\dot v)^2$ where

$E(u,v) = \langle \phi_u, \phi_u \rangle_{\phi(u,v)}$, $F(u,v) = \langle \phi_u, \phi_v \rangle_{\phi(u,v)}$ and $G(u,v) = \langle \phi_v, \phi_v \rangle_{\phi(u,v)}$. In the basis $\{\phi_u, \phi_v\}$ we have $I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$. For any other chart $\psi$ the matrix is given by $(d\sigma)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} d\sigma$ where $\sigma$ is the transition map

**Definition:** The *Gauss map* is the map $N : S \to S^2$ to the 2-sphere defined as the smooth map of unit normal vectors, $p \mapsto N(p)$. This is well-defined if $S$ is oriented.
Locally we can express this as the normalised vector product of $\phi_u$ and $\phi_v$
If $S = F^{-1}(0)$ then $N(p) = \frac{\nabla F}{||\nabla F||}$

**Definition:** The *second fundamental form* on $S$ at $p$ is the quadratic form $II_p(w) = -\langle dN_p(w), w \rangle$ on $T_pS$
Locally we have $II_p(\dot{\alpha}(0)) = e(\dot{u})^2 + 2f\dot{u}\dot{v} + g(\dot{v})^2$ where $e = -\langle N_u, \phi_u \rangle = \langle N, \phi_{uu} \rangle$, $f = -\langle N_v, \phi_u \rangle = \langle N, \phi_{uv} \rangle = -\langle N_u, \phi_v \rangle$ and $g = -\langle N_v, \phi_v \rangle = \langle N, \phi_{vv} \rangle$

**Lemma:** The derivative of the Gauss map, $dN_p$ is self-adjoint.
Indeed the two fundamental forms are symmetric and $II_p(v,w) = -I_p(dN_p(v), w)$

Thus we can diagonalise $dN_p$, so for some orthonormal basis $\{e_1, e_2\}$ of $T_pS$ have $dN_p(e_i) = -k_i e_i$ with $k_1 \geq k_2$. These numbers are the *principal curvatures* at $p$. These are extrinsic curvatures; if we take a piece of paper these are both 0 at any point, but bending to form a cylinder means one direction is now positive whilst the other is still 0. There is an intrinsic notion of curvature:

**Definition:** The *Gaussian curvature* at $p$ is $K(p) = \det(dN_p) = k_1 k_2$
Locally we can write $K = \frac{eg-f^2}{EG-F^2}$, the ratio of determinants of the quadratic forms above - the ratio means the determinant of the transition map cancels, hence this doesn't depend on choice of parameterisation.
There are other equivalences of $K$ such as the limiting area of circles or as -3 times the second derivative of the ratio $\frac{c}{2\pi r}$ at $r = 0$ where $c$ is the circumference of the circle at $p$, see the Bertrand-Diguet-Puiseux theorem.

**Theorem** (*Theorema Egregium*): The Gauss curvature is an isometry invariant!
The proof is some (clever) algebraic manipulations to write $K$ in terms of the fundamental forms and their derivatives, first doing so for the *Christoffel symbols.*

For any point $p$, we have $K(p) > 0$ (*elliptic* - $S$ is on one side of $T_pS$), $K(p) < 0$ (*hyperbolic* - $S$ intersects both sides of $T_pS$, or $K(p) = 0$ (*parabolic*).
We can try to find all the surfaces of constant curvature, by dilation it is enough to consider when $K \in \{-1, 0, 1\}$.
The plane $\mathbb{R}^2$ has constant curvature 0, the unit sphere $S^2$ has constant curvature 1. Can we find a surface of constant curvature -1?

For a geodesic $\gamma$ through $p$ we can find the unique geodesic $\gamma_v(u)$ such that $\gamma_v(0) = \gamma(v)$ and $\gamma_v'(0)$ is orthogonal to $\gamma'(v)$ then define the local parameterisation $\phi(u,v) = \gamma_v(u)$. Coordinates built this way are called *geodesic normal coordinates* and the first fundamental form is of the shape $du^2 + G(u,v)dv^2$ with $G(0,v) = 1, G_u(0,v) = 0$, and the Gaussian curvature is $K = \frac{-\sqrt{G}_{uu}}{\sqrt{G}}$. Using this we have

**Proposition:** For a surface $S$ of constant curvature

(i) If $K = 0$ then $S$ is locally isometric to $(\mathbb{R}^2, du^2 + dv^2)$

(ii) If $K = 1$ then $S$ is locally isometric to $(S^2, du^2 + \cos^2 u \ dv^2)$

(iii) If $K = -1$ then $S$ has local metric $du^2 + \cosh^2 u \ dv^2$

In the third case, with change of variables $x = e^v \tanh(u)$ and $y = e^v \text{sech}(u)$ we get the metric $\frac{dx^2+dy^2}{y^2}$, this will turn out to be the standard parameterisation of the first fundamental form on the upper half plane model of hyperbolic geometry. We know all too well basic results in Euclidean geometry, things like angles in a triangle, areas of shapes, cosine rules and sine laws, but how do these change in spherical or hyperbolic geometry? These geometries, to first order, should be Euclidean so we may retrieve standard results in limiting cases, but naturally may differ on a global scale, let's investigate. We refer to geodesics as lines, the results are as follows.

**Euclidean geometry ($E^2$) :** The standard or flat metric on the plane is $du^2 + dv^2$. The geodesics are segments of straight lines and two lines either intersect at a point or don't intersect (parallel). The isometry group $\text{Isom}(E^2)$ is generated by translations, rotations and reflections, in fact it is easy to see a rotation can be written a composition of two reflections (e.g. rotation by $\theta$ is a reflection in the $x$-axis followed by reflection in a line of angle $\frac{\theta}{2}$, similarly a translation can also be written as a composition of two reflections, so the group is in fact just generated by reflections. The angles in a triangle add up to $\pi$, and for a triangle with sides $a, b, c$ with opposite angles $A, B, C$ respectively we have the *cosine rule*: $c^2 = a^2 + b^2 - 2ab \cos C$ and cyclic permutations, the *law of sines*: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

**Example:** We of course have $E^2$ as a Euclidean manifold. As for a compact one, we have the flat torus: tile $E^2$ in squares and quotient out by the isometries generated by unit horizontal and vertical translations, topologically this is just $\mathbb{R}^2/\mathbb{Z}^2$ but we inherit the Euclidean metric on this geometric torus, this also computes its fundamental group to be $\mathbb{Z}^2$. We can think of the lattice $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ this way, with fundamental domain given by the square with vertices at (0,0), (0,1), (1,0) and (1,1). If you were an ant in this domain facing north you would see the back of yourself, the left body to the right and so on.
In fact we can consider the action of any lattice in $\mathbb{R}^2$, by rescaling and applying

other Euclidean isometries we can take the lattice to be $\Lambda = \langle 1, \tau \rangle$ for some $\tau$ in the upper half space. Two such, do not have equivalent metrics, say by considering the lengths of geodesics, unless they are in the same orbit of the action of $\mathrm{SL}(2, \mathbb{Z})$. We can also see this by carefully pushing the conformal structure of $\mathbb{C}$ onto $\mathbb{C}/\Lambda$, and these give complex tori up to conformal equivalence, with two lattices being equivalent if and only if they scalars of each other, that is their generators are related by a transformation of $\mathrm{SL}(2, \mathbb{Z})$ . The moduli space of metrics on such tori is thus given by the quotient of the upper half plane by the action of the matrix group $\mathrm{SL}(2, \mathbb{Z})$.

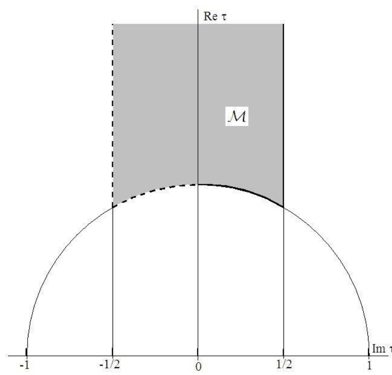



Figure 11: The moduli space of complex tori, figure by [11]

However, this quotient space is not a manifold, there are two pointy bits, given by the very symmetric lattices $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{-3}]$, indeed the upper half space quotiented out by the integer linear fractional transformations $\{z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{Z}\}$ can be identified with the domain $\{z \in \mathbb{C} : \mathcal{I}\mathrm{m}(z) > 0, -\frac{1}{2} \leq \mathcal{R}\mathrm{e}(z) \leq \frac{1}{2}, |z| \geq 1\}$ with the identification of the (half) real lines and the circle $|z| = 1$ by $-x + iy \sim x + iy$ from which we see only $i$ and $e^{\frac{\pi i}{3}}$ do not have a neighbourhood homeomorphic to a disk. The stabiliser of the first is $\mathrm{C}_2$ generated by $z \mapsto -\frac{1}{z}$ and the latter is $\mathrm{C}_3$ generated by $z \mapsto 1 - \frac{1}{z}$. In general for a manifold we can quotient a cover of open sets $U_\alpha$ by groups of diffeomorphisms $G_\alpha$ in a compatible way, this is called an *orbifold.* For a group $G$ acting discretely on a manifold, the points have a neighbourhood of the form $\mathbb{R}^d/G_x$ where $G_x$ is the stabiliser and we can define the *orbifold Euler characteristic* to be $\sum \frac{\chi(O_\mathrm{k})}{|\mathrm{k}|}$ where $O_G$ is the set of points with stabiliser of order k - this is defined so that it is multiplicative as in the usual manifold case. In the above example, the points with trivial stabiliser is the 2-sphere minus 3 points, hence the Euler characteristic of the orbifold is $-1 + \frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$.

**Spherical geometry ($S^2$) :** The spherical metric on the 2-sphere is $du^2 + \cos^2 u \; dv^2$. The geodesics are arcs of great circles and two lines intersect at antipodal points. The isometry group $\mathrm{Isom}(S^2)$ can best be represented by the orthogonal group, $O(3)$ - this is the stabiliser of a point in Euclidean space. The angles in a triangle add up to more than $\pi$, the difference is the area - i.e area($\Delta$)= $A+B+c-\pi$ . We have the *cosine rule*: $\cos c = \cos a \cos b + \sin a \sin b \cos C$ and cyclic permutations, the *law of sines*: $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$

**Example:** $S^2$ is obviously a compact spherical manifold, and we can quotient out by the action of the antipodal map $x \mapsto -x$ to obtain $\mathbb{RP}^2$.

**Hyperbolic geometry ($H^2$) :** The hyperbolic metric on the upper half plane $\{z \in \mathbb{C} : \mathcal{I}\mathrm{m}(z) > 0\}$ is $\frac{du^2+dv^2}{v^2}$. The geodesics are segments of vertical lines or arcs of circles orthogonal to the boundary $\{v = 0\}$ of $H^2$ and two lines either intersect at a point, or don't intersect in $H^2$ but do on $\partial H^2$ (parallel), or don't intersect in either place (ultra parallel). The orientation preserving isometry group $\mathrm{Isom}^+(H^2)$ is the Möbius group preserving the upper half plane, analytically these are maps of the form $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{R}$ and $ad-bc \neq 0$ - this group coincides with $\mathrm{PSL}(2, \mathbb{R})$. The full isometry group is generated by inversions, but since any Möbius map is the composition of two inversions we really only need to understand the orientation preserving ones. The angles in a triangle add up to less than $\pi$, the difference being the area - i.e area($\Delta$)= $\pi - A - B - C$. We have the *cosine rule*: $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$ and cyclic permutations, the *law of sines*: $\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}$
Using the Möbius map $z \mapsto \frac{z-1}{z+1}$ (which is the map that says $z$ is closer to 1 than it is to -1 if and only if $z$ is in the upper half plane) we can transfer this geometry to the unit disk, $\mathbb{D}$. Here we get the metric is $\frac{4}{(1-u^2-v^2)^2}(du^2+dv^2) = \frac{4}{1-|z|^2}|dz^2|$. The geodesics are arc of circles in $\mathbb{D}$ orthogonal to $\mathbb{D}$ - this includes segments of diameters. The isometry group is the Möbius group preserving the disk, $\mathrm{Isom}(\mathbb{D}) = \mathrm{M\ddot{o}b}(\mathbb{D}) = \{z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z} : |a| < 1, \theta \in \mathbb{R}\}$

In the spherical case, to derive the metric one can compute the first fundamental form by viewing $S^2$ as the vanishing locus of $x^2 + y^2 + z^2 - 1$. To see arcs of great circles are geodesics, either one solves the geodesic (differential) equations, or can use a symmetry argument - if a plane intersects a surface with reflective symmetry in the plane, the intersection is a geodesic. This comes from an equivalent characterisation that geodesics are the straightest, that is the second derivative $\ddot{\gamma}(t)$ is normal to the surface: in the case of the sphere this is clear as the acceleration of a circular orbit is towards the centre (in the direction of the radius), which is normal to the surface. A proof of the spherical formulae comes out computationally by considering the matrix $V$ of the vectors given by the vertices of the triangle and the matrix $W$ of the dual basis; the identity $W^T V = I$ gives $W^T W = (V^T V)^{-1}$ and comparing the entries gives the cosine rule, by noting that the angle at a vertex is the angle between the planes spanned the vector at the vertex and by another, which is the complement (w.r.t to $\pi$) of the angle between their normals - the dual vectors.
Let's now turn to hyperbolic geometry, I will give details of the two dimension case but most will generalise to higher dimensions. There are three models of

hyperbolic geometry that we will study - the Poincaré disk, the upper half plane and the hyperboloid model, I might briefly mention one more but won't really use it. First let's define inversion in a circle, it will be easier to think about this on the 2-sphere and project to the plane via stereographic projection.

**Definition:** Take the unit sphere $S^2$ in $\mathbb{R}^3$ and consider the $x$-$y$ plane intersecting $S^2$ in its equatorial disk. Stereographic projection of $p \in S^2$ is given by the intersection of the straight line through $p$ and the north pole, (0,0,1) $= \infty$, with the plane.
Thus we may regard $S^2$ as $\mathbb{R}^2 \cup \{\infty\}$ as the map can be given in coordinates as $(x, y, z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z})$ with inverse $(u, v) \mapsto (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$, this generalises - $S^n$ is homeomorphic to the one-point compactification of $\mathbb{R}^n$ via stereographic projection. This map is conformal, and sends circles to circles and lines; specifically circles through $\infty$ get sent to lines. Möbius maps also take circles to circles on $S^2$.

**Definition:** Given a circle $\Gamma$, inversion in $\Gamma$ is defined by $T \circ \bar{T}$, where $T$ is the Möbius map taking $\Gamma$ to the equator on $S^2$ - the real line in the complex plane. We will work with inversion on the plane unless otherwise stated.
In other words, we take the circle in the plane to the real line, reflect along this, and take this line back to the circle. One can check that this is given by the map $z \mapsto a + \frac{r^2}{\bar{z}-\bar{a}}$ if $\Gamma$ is centered $a$ with radius $r$. From this we can prove the following general facts in the case $n = 2$:

**Proposition:** Let $S \subset E^2$ be a circle in Euclidean space with center $O$ and radius $r$

(i) Inversion in $S$ is the unique map from the complement of the center of $S$ into itself taking circles orthogonal to $S$ to themselves, fixes points of $S$ and exchanges the interior and exterior of $S$

(ii) The image of a point $P \neq O$ is the point $P'$ on the ray $OP$ such that $OP \cdot OP' = r^2$

(iii) The center $O$ of $S$ is taken to $\infty$ in the one point compactification of $E^2$

(iv) Inversion is conformal

(v) Inversion takes circles to circle-lines

(vi) Inversion in a line (regarded as a circle through $\infty$) is just reflection

(vii) A Möbius map is a composition of two inversions

This is easy to see because of the formula, the nature of the wording means it is easy to generalise but for higher dimensions We take (i) to be a definition, and prove the rest elementarily by induction. We can also generalise the Möbius group by taking (vii) to be a definition - see section 2.2. Now let's motivate and derive the results we saw earlier for hyperbolic geometry.

**Poincaré Disk:** Consider the open unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 : ||x|| < 1\}$ with boundary $\partial\mathbb{D} = S^1$. The data given by the isometry group, the geodesics and the metric of hyperbolic geometry has the neat property that they essentially form equivalent characterisations: given the metric we can determine the hyperbolic lines from the geodesic equations, given the isometry group we can characterise the the hyperbolic lines using fixed points, and given the hyperbolic lines we can find the isometry group as the diffeomorphisms sending hyperbolic lines to hyperbolic lines. We will find the metric by asking for hyperbolic reflections (inversions in circles) to preserve hyperbolic distance (we will use freely the properties of Möbius maps discussed above).
Consider a hyperbolic line $L$ and any orthogonal hyperbolic line $M$ in $\mathbb{D}$ (i.e. circles orthogonal to $\partial\mathbb{D}$ that intersect orthogonally), and a vector $v$ emanating from $L$ in the direction of $M$. Consider the Euclidean circle $C$ going through the boundary points $L \cap \partial\mathbb{D}$ and the tip of $v$. Since this is also orthogonal to $M$ (easiest seen by say moving the intersection of $L$ and $M$ to the origin), corresponding points on $C$ on different sides of $M$ must be the same distance from $L$, by varying $M$ - the lines orthogonal to $L$ - we see that $C$ is in fact a equidistant curve, that is it is a hyperbolic cylinder (the lines in the picture would represent the slices of the cylinder being of equal radius, of course this is two-dimensional). The hyperbolic length of $v$is in fact a function of the angle at the tip of the banana since inversions are conformal and we want them to be isometries. We further ask, that to first order hyperbolic lengths should be Euclidean - this means the length of $v$ as a function of the angle $\alpha$ should have finite derivative at $\alpha = 0$, we take this to be 1 (this is so the constant curvature is -1). By a Möbius map we may assume M is a diameter and $v$ is orthogonal to a diameter, and it's base point is a Euclidean distance of $r$ from the origin. In fact to simplify calculations consider let $v$ lie on the positive real axis and then consider the map $z \mapsto i\frac{1+z}{1-z}$ - see figure 12,

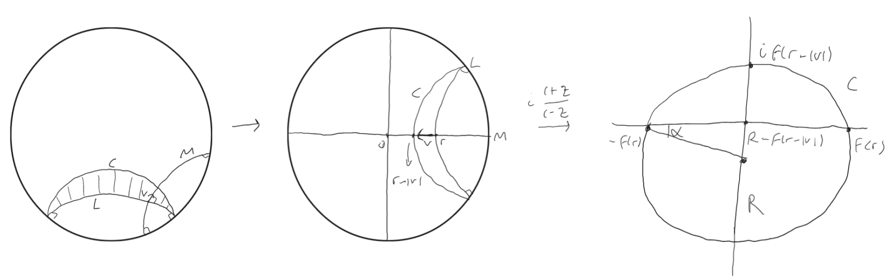



Figure 12: Enter Caption

The angle between $C$ (or rather the image of $C$ under this map) and $L$ is thus $\pi$ minus the angle between $C$ and the real axis. Let $f(r) = \frac{1+r}{1-r}$. From the diagram, by power of a point on the origin, we have

$$f(r-|v|)\cdot(2R-f(r-|v|)=(f(r))^2$$

where R is the radius of $C$ and $|v|$ is the Euclidean length of $v$. From this we get the second line of equality

$$\begin{aligned}2\,\tan(\alpha)&=2\frac{R-f(r-|v|)}{f(r)}\\&=\frac{(f(r))^2-(f(r-|v|))^2}{f(r)f(r-|v|)}\\&=\frac{f(r)-f(r-|v|)}{-|v|}\cdot\frac{f(r)+f(r-|v|)}{f(r)f(r-|v|)}|v|\end{aligned}$$

So in the limit as $|v|\to 0$ and using a (very) small angle approximation we get

$$\begin{aligned}\alpha&\simeq\frac{1}{2}f'(r)\cdot\frac{2f(r)}{(f(r))^2}|v|\\&=\frac{2}{(1-r)^2}\cdot\frac{1-r}{1+r}|v|\\&=\frac{2}{1-r^2}|v|\end{aligned}$$

Since the Euclidean angle is approximately the Euclidean length, which to first order is the hyperbolic length the hyperbolic metric is given by squaring the above: $ds^2=\frac{4}{(1-|z|^2)^2}d|z|^2$

The map above $z\mapsto i\frac{1+z}{1-z}$ doesn't just change coordinates, but also transfers the geometry - it's inverse is the one mentioned earlier when describing the results of the various geometries. This is in fact a Möbius map, so we can instead derive the metric in the upper half plane. This is easy - our banana (cylinder) can instead be sent to what looks like a Euclidean cone (but still is a hyperbolic cylinder in the $H^2$) by sending one of the vertices to $\infty$. Then the horizontal lines ought to be the same hyperbolic distance, so we get the Euclidean length is approximately $y$ times the angle, hence the metric on the upper half plane is $ds^2=\frac{dx^2+dy^2}{y^2}$ which pulls back to the one above via the map from $\mathbb{D}$ to $H$.

Let's better visualise hyperbolic isometries. To classify these we are going to look at it's fixed points. Let $\alpha\in\mathrm{Isom}^+((H))=\text{Möb}(\mathbb{H})$ and recall the full Möbius group acts transitively on $\mathbb{C}_\infty$.

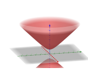
**Definition:** Suppose $\alpha\neq\mathrm{Id}$. Say $\alpha$ is

(i) *elliptic* or a *hyperbolic rotation* if $\alpha$ fixes $p\in\mathbb{H}$. By moving $p$ to the origin of the disk model we recognise this as a Euclidean rotation, so hyperbolic rotations are Euclidean rotations

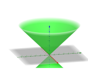
(ii) *parabolic* if $\alpha$ fixes a unique point of $\partial\mathbb{H}$. By sending $p$ to $\infty$ we recognise this as a Euclidean translation in the direction parallel to $\partial\mathbb{H}$

(iii) *hyperbolic* or a *hyperbolic translation* if $\alpha$ fixes 2 points on $\partial\mathbb{H}$. By moving $p$ to $\infty$ we recognise this as Euclidean scaling, so hyperbolic translations are Euclidean scalings

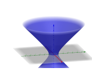
It's an easy check (by say bashing or staring at $S^2$) that all Möbius transformations falls into one of these (mutually exclusive) cases.
We will now look at the hyperboloid model of hyperbolic geometry. Recall that $S^2$ is the unit sphere in $E^3$, that is with the Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$, quadratic form $Q^+$ represented by the identity matrix and inner product $Q^+(x) = x^TQ^+x = x_1^2 + x_2^2 + x_3^2$ have $S^2 = \{x \in E^3 : Q^+(x) = 1\}$, its isometry group preserves $Q^+$ - it is the orthogonal group $O(3) = \{A \in \mathrm{GL}(3, \mathbb{R}) : A^TA = I$ of $\mathbb{R}^3$. The sphere of radius $r$ has curvature $\frac{1}{r^2}$, so a sphere of radius $i$ has constant negative curvature. Recall a parameterisation of $S^2$ given by $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos u$ with associated first fundamental form $\begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u \end{pmatrix}$. Heuristically, we want the sin term to become the hyperbolic term sinh (from the proposition of local constant curvature), so making the substitution $u \mapsto -iu$ we get the standard parameterisation of the one-sheeted hyperboloid $\{x^2 + y^2 - z^2 = -1\}$ in $\mathbb{R}^3$, the first fundamental form transforms to $\begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 u \end{pmatrix}$ but there is a 1 in the first coordinate rather than $\cosh^2 u + \sinh^2 u$ in the usual first fundamental form of this surface. However instead of restricting the Euclidean metric $Q^+$ to the tangent plane, if instead we restrict the *Lorentz metric* $ds^2 = dx^2 + dy^2 - dz^2$ on $\mathbb{R}^3$ - called *Lorentz* space, denoted $E^{2,1}$ - we do indeed get the above first fundamental form. Thus, with respect to the indefinite Lorentz metric (denoted $Q^-$), the sphere of imaginary radius is the two-sheeted hyperboloid $H = \{Q^- = -1\}$.



(a) $Q^- = -1$



(b) $Q^- = 0$



(c) $Q^- = 1$

This metric is clearly indefinite, representing $Q^-$ with matrix with 1's down the diagonal except at the last entry where there is a -1, if we define the inner product of $x, y$ to be $x^TQ^-y$, the vectors with length 1 form the one-sheeted hyperboloid, the vectors of length 0 form the cone that bounds this surface and also the two-sheeted hyperboloid - the vectors of length -1. But on $Q^-1 = -1$ the tangents clearly lie in $Q^- = 1$, so in fact restricted to this sheet we get a Riemannian metric! The sphere of imaginary radius shares several properties with $S^2$, for example we know that a vector on $S^2$ is normal to the surface or that we can intersect planes through the origin to get geodesics on $S^2$, the one through $p$ in direction $v$ parameterised by unit speed is $p\cos t + v\sin t$.

**Proposition:** Let $H^+$ be the component of $H$ with $z > 0$, then

(i) If $x \in H^+$, then its tangent space is the orthogonal complement of $x$; $T_xH = x^\perp$ in $E^{2,1}$

(ii) The orthogonal transformations of $E^{2,1}$ (linear transformations of $\mathbb{R}^3$ preserving $Q^-$) is the isometry group of $H$, denoted $O(2,1)$

(iii) A geodesic through $p \in H^+$ in the direction of $v$ is given by the intersection of $H^+$ with the plane through the origin determined by $p$ and $v$. If $v$ has unit length, it is parameterised with unit speed by $p\text{cosh}t + v\text{sinh}t$

The first follows from the inner product and $\Delta F$ matching (up to a scalar) where $F = x^2 + y^2 - z^2 + 1$, recalling that this measures the direction of greatest change, so is naturally orthogonal. The second follows almost identically as in the Euclidean case, and the characterisation of geodesics being the straightest together with the above gives the third.
Let's look at areas of polygons, we will work with those whose edges are geodesics. To start off we will look at *ideal* triangles - one where all it's vertices lie on the boundary of hyperbolic space. Since Möbius maps act 3-transitively on the sphere we can assume the triangle $T$ has vertices are at -1, 1 and $\infty$. It's area is thus

$$\text{area}(T) = \int_{-1}^{1} \int_{\sqrt{(1-x^2)}}^{\infty} \frac{1}{y^2} dy dx = \pi$$

Now if $T$ is a triangle with angles 0,0 $\alpha > 0$ and $A(T)$ it's area then $A$ is clearly a continuous decreasing function in $\alpha$. From above we have

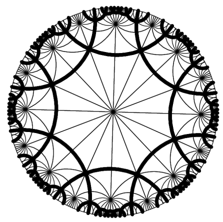
$$A(\alpha) + A(\beta) = A(\alpha + \beta) + \pi$$

and so for $F(\alpha) = \pi - A(\alpha)$ have

$$F(\alpha) + F(\beta) = F(\alpha + \beta)$$

hence $F$ is linear so $A(\alpha) = \pi - \lambda\alpha$ for some $\lambda \in \mathbb{R}_+$, we deduce $\lambda = 1$ from $A(\alpha) + A(\pi - \alpha) = \pi$. The general case is then just splitting a triangle into smaller ones of the above form, entirely computational, we get the area of a triangle with angles $\alpha, \beta, \gamma$ is $\pi - \alpha - \beta - \gamma$, whence the area of a n-gon is $(n-2)\pi - \sum_{i=1}^{n} \alpha_i$ where $\{\alpha_i\}$ are the internal angles.

We can glue regular polygons just as we did earlier, this time in hyperbolic space. To glue by isometries we need the angles at a vertex to add up to to $2\pi$, first consider an ideal 4g-gon for g $\geq 2$ in the disk model - so its vertices are on $\partial\mathbb{D}$. Since the edges intersect the boundary orthogonally, it follows the internal angles are all 0. Shrinking this polygon to a point its area becomes 0, so by above its internal angles are $\frac{4g-2}{4g}\pi > \frac{2\pi}{4g}$ so by continuity at some point we get a lovely regular 4g-gon with internal angles $\frac{\pi}{2g}$. So we can put a hyperbolic structure on a surface of genus 2 by quotienting out $\mathbb{D}$ by the hyperbolic isometries generated



(a) Octagonal tiling of $\mathbb{D}$

(b) A tessellation by M.C.Escher

by the appropriate identification of the above 4g-gon.

These hyperbolic objects don't have a constant curvature, and the ones we have seen have a boundary at we don't include. In fact one cannot find a complete hyperbolic surface:

**Theorem** (*Hilbert*): There is no complete smooth surface in $\mathbb{R}^3$ with a hyperbolic structure!

## 1.4 Geometric classification of 2-manifolds

Every smooth compact 2-manifold can be given a Riemannian metric, indeed we can pull back the Euclidean metric using charts which induces a metric locally, choosing a partition of unity we can average this continuously to build our Riemannian metric. If we scale a surface by say $\lambda$ the Gaussian curvature transforms by $\frac{1}{\lambda^2}$, so we may dilate bits of our surface to ensure it has constant curvature. We have seen few examples of Euclidean and spherical manifolds, but a huge number of hyperbolic ones, could we perhaps put a flat structure on some of these higher genus surfaces? In two dimensions the answer is no:

**Theorem** (*Gauss-Bonnet*): $\Sigma$ compact smooth surface endowed with a Riemannian metric, have $\int_\Sigma K_\Sigma \, \mathrm{dA} = 2\pi\chi(\Sigma)$

This is quite remarkable, the total curvature is a topological invariant! Using this we can see by considering the sign of the right hand side (and taking the curvature to be constant) that on any smooth surface we can have one and only one geometry. To make things clearer for the 3 dimensional case it will be useful to term the three simply connected examples of geometry to be model geometries, from which we can get any other geometry on a manifold. First recall some theory of Riemann surfaces:

**Proposition:** Let G be a group acting freely and properly discontinuously on a Riemann surface R by conformal equivalences. Then the quotient G\R is a Riemann surface and the quotient map is an analytic regular covering map. The complex structure is inherited by (carefully) pushing forward the one on R. Really this is more of a topological statement and so is the proof, the conditions of the action of G are engineered so that the quotient is Hausdorff and the quotient map is a regular covering map, the conformality is simply a technical check. Another topological fact that is true for Riemann surfaces is:

**Proposition:** Every Riemann surface S is conformally equivalent to a quotient G \R where R is a simply connected Riemann surface and G acts on R freely by conformal equivalences. Say S is *uniformised* by R. We then have the beast:

**Theorem** (*Uniformisation*): Every simply connected Riemann surface is conformally equivalent to $\mathbb{C}_\infty, \mathbb{C} or \mathbb{D}$.
This is a remarkable converse to what we have done geometrically above. Also, these are conformally distinct (by compactness and Liouville's theorem). The slightly more useful result from this is:

**Proposition:** For R a Riemann surface,

(i) if R is uniformised by $\mathbb{C}_\infty$ then R is conformally equivalent to $\mathbb{C}_\infty$

(ii) if R is uniformised by $\mathbb{C}$ the R is conformally equivalent to $\mathbb{C}$, $\mathbb{C}^*$ or a complex torus $\mathbb{C}/\Lambda$

So pretty much all Riemann surfaces are uniformised by $\mathbb{D}$; the geometry is hyperbolic. This follows from uniformisation by finding all free subgroups of $\mathrm{Aut}(\mathbb{C}_\infty) = \mathrm{PSL}(2, \mathbb{C})$, and $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b : a \in \mathbb{C}^*, b \in \mathbb{C}\}$

The geometries we saw in 2 dimensions are homogeneous and isotropic: isometries are transitive (on points) and it looks the same from any point - can take any orthonormal frame at a point to any other at any point. These are quite strong properties, and relaxing isotropy in 3 dimensions gives us more geometries. At this point we ponder, what really is a geometry? Most of the time we are just having fun staring off into space sliding down lines, but when on a stroll with a friend you realise life is not all about walking on geodesics. The underlying metric as we have seen plays a crucial role, but so does the group of isometries. It is the interaction of these that make a geometry interesting.

**Definition:** A *model geometry* is a pair $(G, X)$ where $X$ is a connected and simply connected smooth manifold, and $G$ is a Lie group of diffeomorphisms acting transitively on $X$ with compact point stabilisers such that $G$ is not contained in any larger group of diffeomorphisms of $X$ satisfying the condition above. We also assume there exists a compact manifold modelled on $(G, X)$.
The simply connectedness is to obtain as large a space as possible, if $X$ is not we can find a simply connected cover which gives a space that does not come from $X$ but has the same geometry. Similarly $G$ is maximal so we look for subgroups of isometries to quotient by to obtain new spaces. Thus in two dimensions we have:

**Theorem:** The 2 dimensional geometries are Euclidean, spherical and hyperbolic
Indeed we have constant curvature by transitivity, and by rescaling it is 0 or a unit, we are done by uniformisation.

# 2 Geometry of 3-manifolds

We saw with 2-manifolds that we can obtain them by quotienting the action of a subgroup of isometries on a simply connected geometry, these turned out to be isotropic and we can obviously generalise these three spaces, objects whose geometry comes from these are still quite easy to live in. Our pools of geometry that we can dip into and use to get out other geometries will live as a family of eight. The old traditionalist heads of the family at least in the Euclidean and hyperbolic case will generalise, $S^3$ is a little more interesting than its predecessor because unlike $S^2$ it has evolved a group structure. Three dimensions is not as plain, and has more than just this to offer. There are five more younger figures, four of which really come by mixing the two dimensional geometries - the genes of the older generation really have mixed well; $E^3$ can be thought of as stacked by planes (say parallel to the xy-plane) such that any vertical lines passes through all of them at exactly one point, upon quotienting out by a leaf (the z-axis) we get a plane - i.e $E^2 \times E$. Now if we can attach some other two dimensional geometry to such a string we get two pretty straightforward geometries: $\mathbb{H}^2 \times E$ and $S^2 \times E$. Here the strings have curvature 0, if instead they had a unit curvature we get two more geometries; if we just fiber over $S^2$ we get back $S^3$ with (1-d) foliation given by the *Hopf fibration.* Over $E^2$ we get *nilgeometry*, the *Heisenberg group* is a brilliant geometry, and over $\mathbb{H}^2$ we get the geometry of $\widetilde{\mathrm{PSL}}(2,\mathbb{R}) = \widetilde{\mathrm{SL}}(2,\mathbb{R})$ (the tilde means to take the universal cover). Finally we have the member who is slightly different to the rest, perhaps labelled as 'going through a phase', this is *solgeometry.* Whilst I've set up this family as 3 - 4 - 1 which comes from looking at the dimensions of point stabilisers, there is another neat way to view the group dynamic, which will make more sense after studying these geometries in a little more detail.

The classification is not as neat. Previously, the diamond was Euler's number, but this family doesn't prize the artifact - wood is rarer than diamond in the intergalactic market. Indeed if we consider a cell complex for a 3-manifold, then its dual has the same Euler characteristic. But the the sum V - E + F - T where T is the number of tetrahedra, switches sign to T - F + E - V, hence the Euler characteristic is equal to its negative, so it is 0 - this argument generalises to odd dimensions. Another way to see this is to look at the *Heegaard splitting*; if we take two handlebodies and glue their boundaries (by an orientation reversing homeomorphism) we get a 3-manifold, so doing this in reverse we can decompose any 3-manifold into two handlebodies (though this is hard to prove). Now for a cell decomposition, we can take a a little neighbourhood of the two skeleton and similarly for its dual - this gives the Heegaard split, from which we see the Euler characteristics cancel.

Figure 15: Splitting two tetrahedra by thickening 0 and 1-cells for it and its dual

This also means it is hard to look for a uniqueness property of putting a geometry on a structure like we did before using Gauss-Bonnet. Secondly, how we construct the space actually does matter, in some sense our constructions of an object in two dimensions were not really different but in three these methods systematically give objects of a certain geometry - hence the formal approach to certain methods in section 1. However, we can still attack Earth to obtain wood; three is sufficiently small a number that many traits we have in two dimensions still hold or prove useful here, such as foliations - which will essentially give us a classification - or neat results on differentiability, these tend to not work so well in higher dimensions.

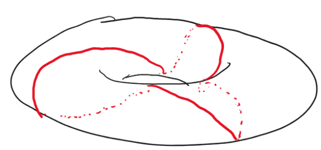


Figure 16: Can foliate a torus with circles, this one is generated by lines of slope $\frac{1}{3}$ on the square

Before looking at the geometries, it'll be helpful to know when we can actually glue the faces of a polyhedron. This wasn't a concern in two dimensions, there were only two homeomorphisms taking edges to edges (up to isotopy) and pairing two edges with an orientation preserving map put a Möbius strip in the manifold making it non-orientable, but either way we got a manifold. Working with linear approximations of surfaces turns out to be more than enough to develop lots of theory, it is no surprise since locally we work with tangent planes. But it is surprising the kind of things we can prove, like we can't squish $S^n$ to look like $S^m$ for $n \neq m$, these results really don't need any sort of smoothness everywhere conditions. Of course, once we have the power of combinatorics it makes a lot of hard work become factory computation. When doing homology a lot of simplicial theory is developed, and it's mostly pages full of symbolic diarrhoea, very little conceptual thinking is required, it'll more or less be a waste to do that again but it is useful to have done it once at some point in life. We consider a cell complex of a surface, the star of a face is a neighbourhood within this complex (so all faces that contain it) and the link is the boundary of this. It is best to think about a small ball around around a vertex, but the ball is linearised according to the complex (this is really just subdividing to make things small enough). If we glue a manifold, a neighbourhood of the vertices ought to look like a ball, the boundary must be $S^2$, if this is the case then its star is a cone of the link, but taking the union of shrinking spheres just gives us a ball. Turns out its enough to consider the neighbourhood of the vertices and not all boundary points First, lets look at the three big personalities. The converse is also true since it must be locally simply connected at the vertex. Thus we have:

**Theorem:** A three dimensional gluing X is a manifold if and only if its Euler characteristic is zero. In fact is X has k vertices, $\chi(X) = k - \frac{1}{2}\sum_1^k \chi(\text{link}(v_i))$ This follows immediately from noting the sum is 2e-3f+4t - as every edge will intersect the links around a vertex for precisely two of them, contributing two such vertices to the total etc.

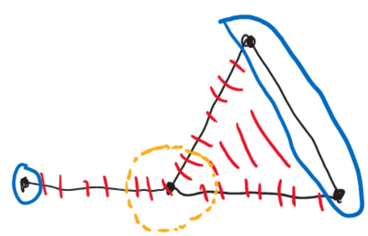


Figure 17: The star (red) and link (blue) of a vertex, really we view its world through the small yellow glass around it

## 2.1 Euclidean space, $E^3$

There is not too much to talk about, we've heard the tales of the old man so very often - we (locally) live in this space and three dimensional Euclidean theory is very closely related to two dimensions. Once again a general isometry of $E^3$ is of the form $x \mapsto Ax + b$, where $A$ is a member of the orthogonal group O(3) and b is any vector (to translate), in other words its the semi-direct product of translations with orthogonal transformations - the point stabilisers. Moreover any element of O(3) can be written as a reflection or a composition of reflections; we can decompose this group by using the index two subgroup SO(3) (the maps with determinant 1 in this group) which are exactly the rotations, combined with any reflection.

A nice compact Euclidean object is the classic doughnut you'll find in any intergalactic bakery - the three-torus. Just as before we tile $\mathbb{R}^3$ with unit cubes and take the quotient by the isometries that glue the boundaries of these cubes in the obvious way: algebraically this is quotienting $\mathbb{R}^3$ by the integer lattice $\mathbb{Z}^3$. This of course can be identified (via the gluing boundary) as $S^1 \times S^1 \times S^1$. Living in this space may spark a (possibly short lived) sense of amusement, upon looking up you see your feet and to an outside observer it looks like those cartoon prison complexes but goes on forever, and the occupants have decided to share one brain and body, one may imagine it would be a perfect way for a single person to feel part of a synchronised dance group like in the movies. And so whilst it may seem like you have a lot of freedom, you don't, at least up to the conundrum of prisons constraining the body but not the mind.

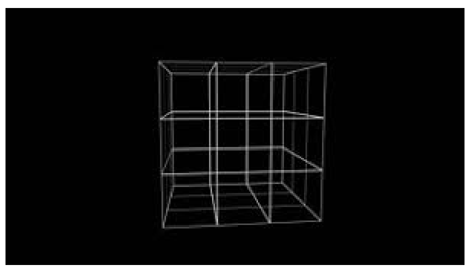


Figure 18: The boxes are identified to give the three-torus

Perhaps one thing to itch your brain; for the one-torus we have a lovely embedding of $S^1$ into $\mathbb{C}$ given by $\mathbb{R} \to \mathbb{C}$, $\theta \mapsto (\sin\theta, \sin'\theta)$ where sin and its derivative satisfy the quadratic relation $\sin^2\theta + \cos^2\theta = 1$. Similarly for the two-torus we have the *Weierstrauss* $\mathfrak{p}$*-function* which (almost) embeds a torus $\mathbb{C}/\Lambda \to \mathbb{C}^2$ via the map $z \mapsto (\mathfrak{p}(z), \mathfrak{p}'(z))$ where the coordinates satisfy a cubic relation. Is it possible the three torus admits a similar (almost) embedding into $\mathbb{C}^3$? Now when God threw their dice it happened that the third dimension wasn't blessed with a complex type structure, so it is a bit hopeful, but maybe the four-torus does since we have access to the quaternions?

## 2.2 Spherical space, $S^3$

This space is a little hard to visualise but spend a bit of time with them and one begins to understand. One way is, as before, to take the one-point compactification of $\mathbb{R}^3$, this way we just need to distort our image of the space around us remembering $\infty$ is also accessible. However, this is not satisfactory. For one it is not as fat as we would like, obviously it's pictorially Euclidean. In two dimensions we could see this space by looking through a glass bottle, if you put a bit of reflective material it becomes even closer (really this would resemble a concave mirror, where as you move from the mirror to the focal point things get large and then hits infinity and then inverts and becomes smaller: this is like sending an object in your field of vision bounded by two great circles and the focal point is like the antipodal - it doesn't quite work perfectly though), but it doesn't really respect the true size of lines - indeed consider a great circle on $S^3$, as we move it closer to $\infty$ it gets bigger and bigger when we stereographically project to $\mathbb{R}^3$.

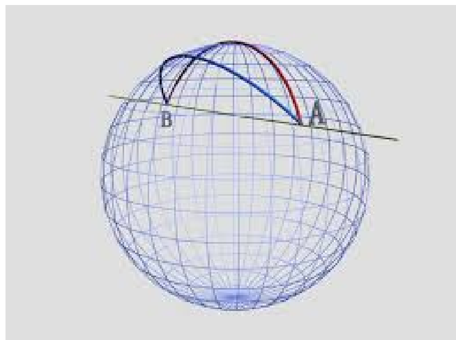



Figure 19: You are at A, your field of vision converges at B and wraps to the back of your head on the 2-sphere

Now $S^3$ naturally sits in $\mathbb{R}^4$ as $\{x_1^2+x_2^2+x_3^2+x_4^2=1\}$ or in $\mathbb{C}^2$ as $\{|z_1|^2+|z_2|^2=1\}$. Nevertheless, we can still do some things with this perspective, if instead we imagine living inside it - for this one gets into the zone, the cloudy ultra-focused tunnel where everything around you becomes fuzzy and you get lost in imagination, the body on autopilot. It is different to $S^2$, though it resemble some bits of $S^1$ .

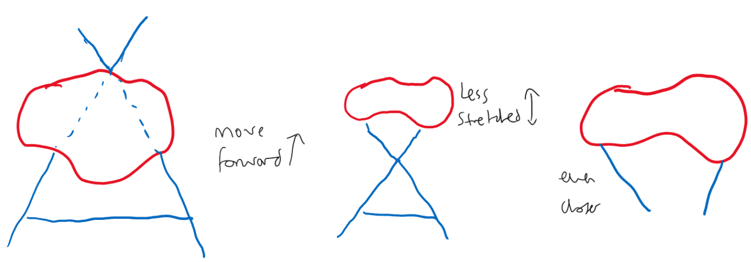



Figure 20: An object far away will appear quite big and stretched but becomes smaller as we approach, until we are close enoguh it is like Euclidean space again

The feeling of $S^2$ was quite rigid you can imagine it as a shell filled with goo and no matter what you do with it there is a needle somewhere poking through, this was best seen using the hairy ball theorem. Maybe another take on this is that there are really only two homeomorphisms on $S^2$ that come from geometries - the identity and antipodal (formally an element of the deck group of two dimensional spherical geometry is a homeomorphism of $S^2$, and if it is orientation preserving it must have a fixed point since it is not homotopic to the antipodal map, so it follows the only two dimensional spherical geometries are itself and $\mathbb{RP}^2$). The point of the elliptic space was that if you are an ant, you would see an object getting further away from to appear bigger once its half way round (your field of vision is two great circles converging to your antipode) and distressingly the ant would start throwing hands at its enemy who is on the other side of the world and likely be sent to an asylum for acting crazy, whereas in elliptic space at least this distance problem would go, you would only have to endure a flipped hazy image of yourself. On the other hand, $S^1$ was not too problematic, we can simply rotate it. Similarly we can multiply any point in $S^3$ by $e^{it}$, $t \in \mathbb{R}$ to see it is rounder, this also shows $\mathbb{RP}^3$ is orientable (the antipodal map is homotopic to the identity by these rotations), though we would still have things appearing close as they went closer to your antipode. In fact this rotation sends the set of complex lines through the origin in $\mathbb{C}^2$ to itself - these intersect $S^3$ in great circles, moreover through any point there is a unique such great circle obtained this way, this is called the *Hopf fibration* and the isometries above leave this invariant.

Visually these move the points along the circles, and the motion looks the same about any point. Formally, we get a map $S^3 \to S^2$ sending the points on a given great circle above to the corresponding complex lines parameterised by $\mathbb{C}_\infty$ - the fibers are $S^1$, so as Thurston put it, $S^3$ is a 2-spheres worth of $S^1$'s. This foliation is not quite canonical as we had a choice of identification of $\mathbb{R}^4$ with $\mathbb{C}^2$, the same way how our foliations of Euclidean space with parallel lines is not canonical. Another interesting observation is that with the standard form of the metric on $S^3$ given by $\mathrm{d}s^2 = \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$ if we push this forward by the map given above we induce the standard metric on $S^2$.

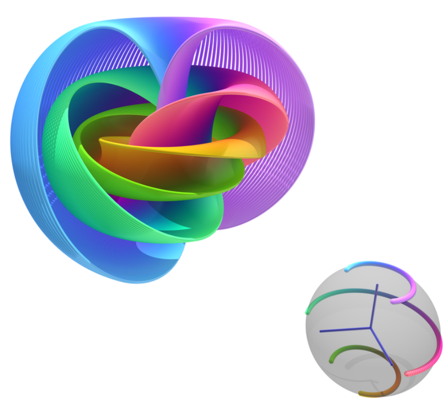


Figure 21: A visualisation of the Hopf-fibration by Niles Johnson, [12]

Looking at its stereographic projection and seeing nested tori cover $\mathbb{R}^3$ except the circle given by the z-axis we recall the pictures we had when moving the torus over infinity, thereby exchanging its loop generators. If glue the two tori, obtained by interchanging the loops we get all of $S^3$, that is we have found a Heegaard splitting of the three-sphere into two solid tori.

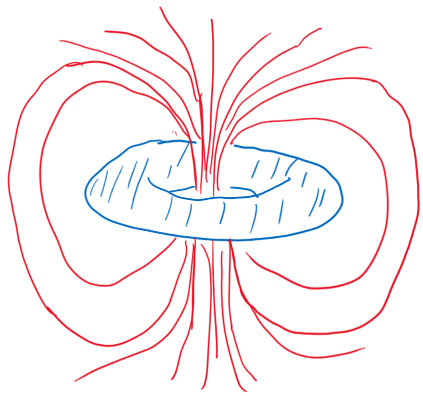


Figure 22: Two solid tori filling $S^3$, though one of its circles goes through $\infty$

For a better study we will make use of the quaternions, $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R},\ ij = k = -ji,\ i^2 = j^2 = k^2 = -1\} = \{z + wj : z, w \in \mathbb{C}\}$. So using the latter we can identify $\mathbb{H}$ with $\left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} : z, w \in \mathbb{C} \right\}$. In other words we can identify the three-sphere with $\mathrm{SU}(2, \mathbb{C})$ as the unit sphere in $\mathbb{H}$. This endows $S^3$ with a group structure with left or right multiplication giving a self action, hence it is a Lie group, the inverse of an element can easily be found since $AA^* = \mathrm{Id}$ where $A^* = \bar{A}^T$ is its Hermitian conjugate. There are some cool things we can do, for example observe that two matrices are equivalent (conjugate by a matrix in $\mathrm{SU}(2, \mathbb{C})$) if and only if they have the same trace - this is seen by diagonalising to the matrix with entries $e^{i\theta}, e^{-i\theta}$ - a point on a *maximal torus*, T - and so the angle is determined (up to sign) by $\frac{1}{2}\mathrm{trA} = \cos\theta$, in other words, excluding the conjugacy classes $\{I\}, \{-I\}$, each ccl is homeomorphic to the two-sphere. Indeed by considering the action of $\mathrm{SU}(2, \mathbb{C})$ on $S^2$ by Möbius maps we see it has stabiliser T (I think for the homeomorphism of this first isomorphism theorem type result a subtle argument is needed, which I will talk a bit about later). But the quotient of $\mathrm{SU}(2, \mathbb{C})$ by T can also be identified with a conjugacy class by noting T is the center of any diagonal element of the form above.

We now wear our algebraic glasses to view the Hopf fibration, first look at the group $V = \langle i, j, k \rangle$ which can be identified with $\mathbb{R}^3$, then for any $v \in V$ with $|v| = 1$ we have the subgroups $S_v = \{e^{\theta v} : \theta \in \mathbb{R}\}$ which is homeomorphic to $S^1$. Then we may consider the quotient $S_i \backslash \mathrm{SU}(2, \mathbb{C})$ which can also be written

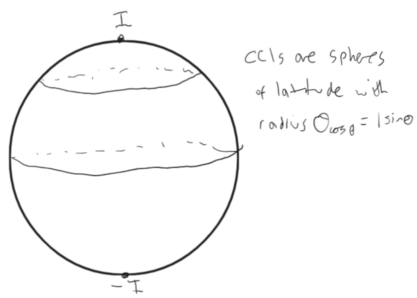



Figure 23: The conjugacy classes in $S^3$ (albeit a dimension lower)

as the quotient of $\{z + wj\} \backslash \sim$ where $(z, w) \sim (e^{i\theta} z, e^{i\theta} w)$ defines the map to $\mathbb{C}_\infty$ by $(z, w) \mapsto \frac{z}{w}$, which is exactly the map given by the Hopf fibration!

The circles in the Hopf fibration are the intersections of special planes with $S^3$. More generally, the intersection of any plane in $\mathbb{R}^4$ with $S^3$ defines a great circle, which are the geodesics and an isometry group can be given as $\mathrm{SU}(2, \mathbb{C})$ by the identification (note conjugation is already in here since $q^{-1} = |q|^{-2}\bar{q}$). The full isometry group, as before is given by the orthogonal group $\mathrm{O}(4)$. There are many other interesting geometric properties of the three sphere but one general question is when can we put a (compatible) group structure on various spheres (respecting their topology)? To answer this we construct the vector field on $S^2$ by considering left multiplication by any element $s \in S^2$, so for any fixed vector $v \in T_e S^2$ the derivative of the maps $x \mapsto sx$ at $v$ give a non-zero vector field, contradicting the hairy ball theorem.
An alternate proof which generalises is to look at the unit tangent bundle $\mathrm{S}(\mathrm{T}S^2)$ $= \{(x, v) : x \in S^2, v \in T_x S^2, ||v|| = 1\}$ which can be identified with $\mathrm{SO}(3)$ via $(x, v) \mapsto (x, v, x \times v$. Now if $S^2$ were a Lie group it can be shown that we can identify this bundle with $S^2 \times S^1$. But $S^3$ is simply connected so its the universal cover of the index two subgroup $\mathrm{SO}(3)$, whence its fundamental group is $\mathbb{Z}/2\mathbb{Z}$, on the other hand the fundamental group of $S^2 \times S^1$ is $\mathbb{Z}$, thus $S^2$ is not a Lie group. It turns out only $S^0, S^1, S^3$ and $S^7$ are Lie groups out of all the spheres.

There are several interesting spherical objects in three dimensions; we can try a more general map than just the antipodal by identifying points given by a rational rotation in $(\mathbb{C}^2)$ coordinates - these give the lens spaces which are quite interesting but I want to focus more on another example - the *Poincaré dodecahedral space*. We glued opposite edges with certain orientations to obtain 2-manifolds, here we glue opposite faces much like we did for the three-torus, but the pentagonal faces don't match up perfectly, so we don't get a Euclidean structure. If you work out the phase shift, then rotating each face by $\frac{1}{10}$ of a full twist we impose a valid identification, this is seen by considering a small neighbourhood of a vertex and making sure the edges glue to give a 3-manifold. A bit of edge chasing shows three edges are identified, so to tile a space with 3 dodecahedra about each edge we need the dihedral angles to add up to $\frac{2\pi}{3}$. This is too big to be hyperbolic or Euclidean (these angles would be $\leq \arctan(-2) < \frac{2\pi}{3}$ so we turn to $S^3$. We try to replicate the argument before to find such a dodecahedron. A super tiny one is basically Euclidean, so we can find one with angles smaller than we'd like. Feeding it loads we fatten it enough to obtain a two-sphere, so the dihedral angles are $\pi$, thus at some point we get the desired angle and we can tile $S^3$, the quotient by the subgroup generated by the isometries corresponding to the identification gives us the space. It turns $S^3$ is tiled with 120 of these (being compact). Interestingly this space has the same homology as the three-sphere (so it's sometimes called the homology sphere), but it's fundamental group is of order 120 (binary icosahedral group), so it is not simply connected, in particular not homeomorphic to $S^3$.

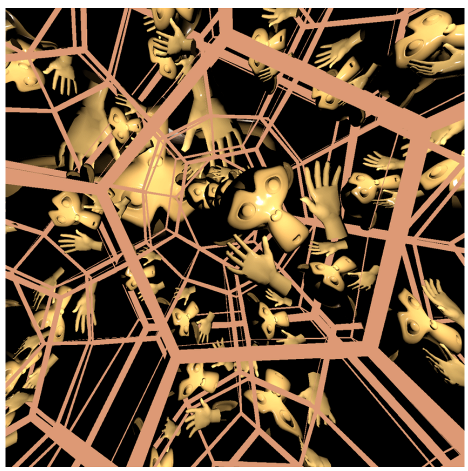


Figure 24: Inside Poincaré dodecahedral space, image by [13]

## 2.3 Hyperbolic space, $\mathbb{H}^3$

Here's the old timer who always has an interesting story to tell, having lived life to its fullest. Like $E^3$, but unlike $S^3$, it is similar to its predecessor, but it is knot the same (sorry). Knot complements will be a fun visualisation from which we can find hyperbolic objects, but first lets generalise some results from the first section. The two main models are the upper half space with metric derived as before by looking at a hyperbolic cylinder (which looks like a banana or cone if one vertex of the banana is sent to infinity) to obtain $\frac{1}{x_0^2}\mathrm{d}x^2$, and the ball $\mathbb{B}^3$, with boundary to be taken as infinity as in the disk case - its metric is given as before in terms of Euclidean distance. The orientation preserving isometry group is taken to be the Möbius group, where each such isometry is defined to be a composition of two hyperbolic reflection. The hyperbolic lines are as before segments of circles or lines orthogonal to the bounding plane $\{x_0 = 0\}$. These are in fact all equivalent characterisations, with the proof as before so it doesn't matter which we start off with. Pretty much most things generalise, for example:

**Proposition:** Let $S \subset E^n$ be a $(n-1)$-sphere in Euclidean space with center $O$ and radius $r$

(i) Inversion in $S$ is the unique map from the complement of the center of $S$ into itself taking spheres orthogonal to $S$ to themselves, fixes points of $S$ and exchanges the interior and exterior of $S$

(ii) The image of a point $P \neq O$ is the point $P'$ on the ray $OP$ such that $OP \cdot OP' = r^2$

(iii) The center $O$ of $S$ is taken to $\infty$ in the one point compactification of $E^n$

(iv) Inversion is conformal

(v) Inversion takes spheres of any dimension to spheres (of the same dimension)

(vi) Inversion in a plane (regarded as a sphere through $\infty$) is just reflection

So I'm not going to relist all the theory developed (albeit with a slight change in the numbers). Hyperbolic three-space is a little more interesting, so lets better understand the orientation preserving isometries, in fact there isn't really much that's new. In two dimensions we could classify them by fixed points, some on the boundary. Here we do a similar thing, this time they will have fixed points but also an axis running through them (unless there's just one fixed point on the boundary). It is important to remember that the boundary is not part of hyperbolic space, it is at infinity, and so we can send a point on this to the infinity in the sense of the one point compactification of the boundary (we did this freely in two dimensions as the disk model doesn't have a favourite point on its boundary). First observe that an axis must be unique, otherwise it would either fix them both pointwise and thus a plane spanned by them and so must be the identity, or it translates along a line but this would mean the distance between the axes is bounded (by periodicity and spamming the isometry), this is absurd by travelling along the lines towards infinity. The actual classification looks exactly like it did before, but with an axis:

**Definition:** Suppose $\alpha \neq \mathrm{Id}$ is a orientation preserving isometry of hyperbolic space. Say $\alpha$ is

(i) *elliptic* or a *hyperbolic rotation* if $\alpha$ fixes $p \in \mathbb{H}^3$. By moving the axis so it becomes the (positive) z-axis we recognise this as a Euclidean rotation, so hyperbolic rotations are Euclidean rotations. The smallest distance obtained between a point and its image is of course 0, the happens on the axis of rotation, other points are moved in a hyperbolic circle (which is also a Euclidean circle) around the axis

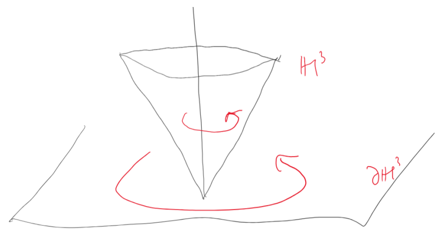



Figure 25: Hyperbolic rotation about fixed axis

(ii) *hyperbolic* or a *hyperbolic translation* if $\alpha$ fixes 2 points on $\partial\mathbb{H}^3$, this translates the axis along itself. By moving the axis as above we recognise this as Euclidean scaling, so hyperbolic translations are Euclidean scalings. Points are moved away from 0 (towards infinity), and the smallest distance moved (inside hyperbolic space) is positive and attained exactly on the axis

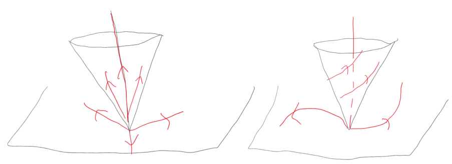


Figure 26: Hyperbolic translation (left) and mixed with rotation (right) called *loxodromic*

(iii) *parabolic* otherwise. This fixes a unique point on the boundary and by to $\infty$ on the sphere we recognise this as a Euclidean translation in the direction parallel to the bounding plane. Here the smallest distance is not attained, but you can find points with arbitrarily small distance (from its image) - simply move along the planes parallel to the bounding one

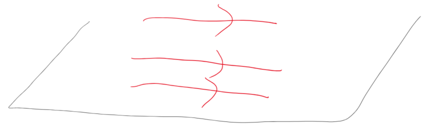


Figure 27: Parabolic transformation, with action on a horosphere

The classification using distances and their infimums generalises to $\mathbb{H}^n$. We can compute the dimension of the Möbius group: there are $n$ dimensions from translations, and the stabiliser of a point has dimension that of the orthogonal group O(n) which we calculated in the previous section, this dim(Möb($\mathbb{H}^{\ltimes}$) = $\binom{n}{2} + n = \binom{n+1}{2}$. In three dimensions we get 6, which can also be done by identifying this group with PGL$(n, \mathbb{C})$ for $n = 2$ which has dimension $2(n^2 - 1)$ - $n^2$ comes from GL$(n, \mathbb{C})$, the minus one from fixing its determinant (see preimage theorem in previous section; apply rank nullity to tangent planes), and then the factor of two comes from going to an $\mathbb{R}$ vector space. In higher dimensions, these groups clearly don't coincide (by e.g. dimension). This also works for $n = 2$ since we change the $\mathbb{C}$ into $\mathbb{R}$ so the factor of two drops.

Let's look at some hyperbolic objects, the first to try is what we did before - to glue polyhedra which tile the space. In fact we can turn to dodecahedron just as in the spherical case, this time we rotate each face by a factor of $\frac{3}{10}$ of a clockwise twist. The resulting space is the *Seifert-Weber dodecahedral space*, and chasing shows there are 5 edges identified at a time. So the desired dihedral angle is $\frac{2\pi}{5}$. The limiting case is when we blow up this dodecahedron so that its vertices touch the boundary, easy to visualise in the ball model. The three faces which meet at a vertex intersect each other at angle equal to intersection of the 3 circles that are given by the intersection of the planes with the boundary. By symmetry (and stereographic projection) we see these form an equilateral triangle, whose angles are far less than $\frac{2\pi}{5}$, so repeating a similar argument we can tile this space with appropriate dodecahedra and quotient out.

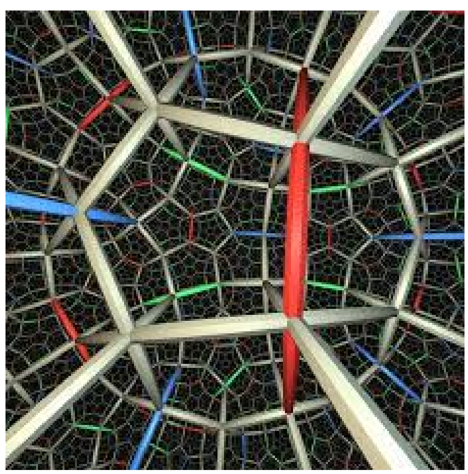


Figure 28: Seifert-Weber space, produced by the Geometry Center

An interesting construction is to use knot complements. We work in $S^3$, though it is easier to visualise in $\mathbb{R}^3$ The example of a figure eight knot complement took me a bit to understand so I'm going to draw a few pictures to show what is going on. First consider the figure eight knot. The complement of this space will be some 3-manifold, the faces will be attached to the actual loop (which are removed), so it will be easier to construct the faces as attached to the loop. To do so we add two extra 1-cells as shown. Now we attach 4 two cells as shown, it is easiest to imagine them as pieces of paper which have been cut allowing for twists.

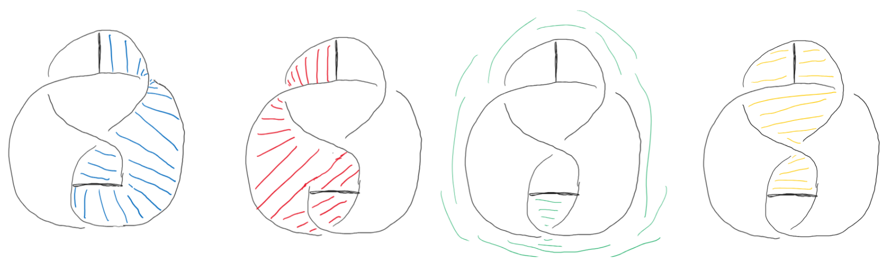


Figure 29: Four two cells, in the middle the red goes above the blue on top, and it goes under the blue below

The complement of this space now leaves two components, as either sides of the faces cannot be reach by a tiny person without crossing through them. Thus, we have identifications on two tetrahedra, the faces representing each regions which are in the same component. Now squish the edges which correspond to the original loop (not with the added 1-cells), to get a different picture of the identification, we can glue the outsides to see we are gluing two adjacent tetrahedra (with their vertices removed). This is easier drawn by stereographically projecting from one vertex:

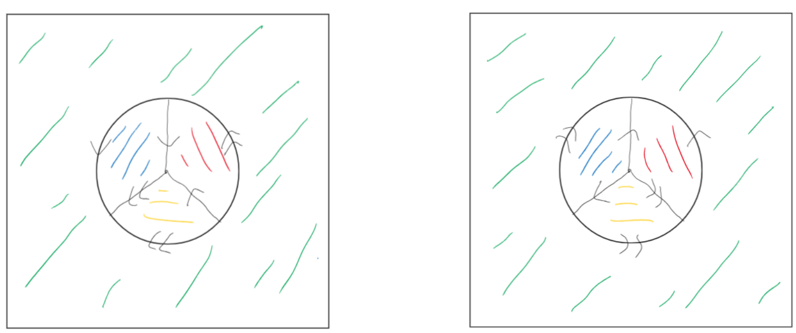


Figure 30: The gluing conditions of the two tetrahedra, the arrows represent the added edges which need to be glued

We are basically done now, all we have to do is put on the hyperbolic structure using a similar argument as before. Since the tetrahedra don't actually have vertices, we can banish them to the cold fringe - the boundary (of the ball model).

In the Klein model one sees two legitimate Euclidean adjacent tetrahedra. We must check the dihedral angles add up consistently; chasing shows there are 6 edges identified at a time, but an ideal tetrahedron has dihedral angles of $\frac{\pi}{6}$, just what we need! To see this the intersection of three symmetrically placed planes on the boundary have angle intersections equal to that of the circles that they form by intersecting the boundary, placing the intersection of these (the vertex) at infinity and stereographically projecting we get an equilateral triangle. Now we just consider the hyperbolic isometries which give the identification, and quotient out $\mathbb{H}^3$ by the group generated by them, which is the fundamental group of the hyperbolic three-manifold we now have constructed, sweet! It is interesting to note, if we did put back the vertices their link forms a torus (by the formula given earlier). In fact this is a general phenomenon, if we take a complement of a tubular neighbourhood of a knot (in $S^3$), we get a three manifold with torus boundary - the boundary of the tubular neighbourhood is clearly a torus; it's inside (bounded component of complement) is a solid torus (this way we obtain $S^3$ as the union of two solid tori in the case our knot is the *unknot*, which is easiest seen by taking it to be the z-axis).

### 2.4 The other geometries

The ancients we studied looked the same every where. But the youth have the experience of selfies, so know the direction in which we view is important. These geometries have the property that they are *anisotropic*, we will see the geodesics appear different depending on the direction we are looking in. This will be relatively brief, certainly the two boring members of the family that follow all the rules, $S^2 \times E$ and $\mathbb{H}^2 \times E$ are more or less self-explanatory in their geometry. It is easiest to visualise it as climbing a pole ($E$) and looking around you with your spherical or hyperbolic glasses we developed in the previous section - perhaps it is more natural to think of say $S^2 \times S^1$ (and then take the universal cover to unwrap the loops) where the second factor is the angle you rotate. Both of these have 1 dimensional point stabilisers, with the orthogonal group O(2) having index two in this (coming from the two manifold, and allowing for reflections orthogonal to this - in line with $E$ gives the rest of the stabiliser). The group of isometries can be built in a similar way e.g. in the first case we get O(2) $\times \mathbb{R} \times C_2$, for the latter we replace the first factor by the Möbius group. It is also simple to construct compact manifolds with such geometry; such as $S^2 \times S^1$ for the former or $X \times S^1$ where $X$ is any two-dimensional compact hyperbolic object, for the latter.

Now we interact with that cool member that turns events into a blast and is generally there to have a good time. Sometimes known as twisted $E \times \mathbb{R}$, the *Heisenberg group* can be defined as $\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R}^3 \right\}$ with matrix multiplication, this is clearly isomorphic to the vector space $\mathbb{R}^3$. It is easy to see it is non-abelian by multiplying out. In fact it is a non-abelian nilpo-

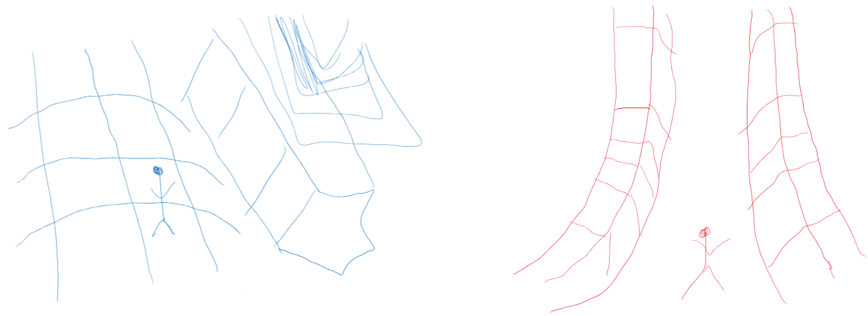


Figure 31: Blue is $\mathbb{H}^2 \times \mathbb{R}$, here things get more packed as they ascend vertically, and very small far in front. Red is $S^2 \times \mathbb{R}$, where things get stretched and say go upwards and appear larger as you look ahead

tent group; very close to being abelian. Indeed, write $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ as generators, then $C = [A, B] = ABA^{-1}B^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ generates the center of group (as it commutes with $A$ and $B$, the rest is symbolic algebra), but quotienting out by this we forget what's in the top right entry and so it becomes abelian.

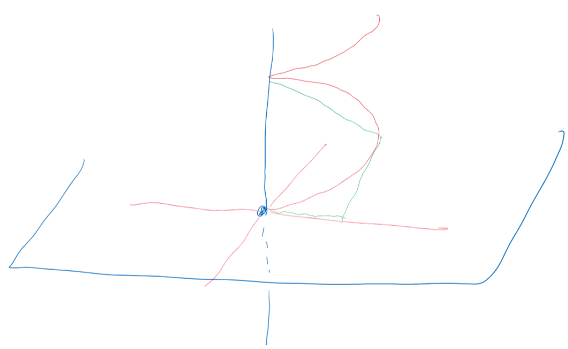


Figure 32: A view of the Heisenberg group, the square translates you up

Geometrically, the fact that $C$ is the commutator of $A, B$ shows that moving right then forward, then left then backward gets you to the same place in the plane but one unit up - we are spiralling upwards as we walk in squares. To understand this intuitively, we see that if we consider $\exp\left\{\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$, then we induce the group structure on $\mathbb{R}^3$ by $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx'))$ and the area of the triangle with vertices $(0, 0), (x, y), (x + x', y + y')$ is exactly $\frac{1}{2}(xy' - yx')$ (half the vector product). This means the group actually, in a sense measures area, so for above we are asking what is the area of formed by the two triangles that split the unit square along the positive diagonal - this is clearly 1!.

As an aside, if we consider a curve $\gamma(t) = \exp\left\{\begin{pmatrix} 1 & x(t) & z(t) \\ 0 & 1 & y(t) \\ 0 & 0 & 1 \end{pmatrix}\right\}$, then by raw computation we have $\gamma'(t) = v(t)\gamma(t)$ for some smooth $v$. Noting that the matrix we take the exponential of, vanishes when raised to the third power, so using the series of the exponential we can compute $a = x, b = y$ and $c = z + \frac{1}{2}xy$, from which we get that $v = \begin{pmatrix} 1 & x' & z' - \frac{1}{2}(xy' - yx') \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix}$. A special case is when the top right is 0, that is $z' = \frac{1}{2}(xy' - yx')$ - a curve $\gamma$ with satisfying this is called a *Carnot curve*.

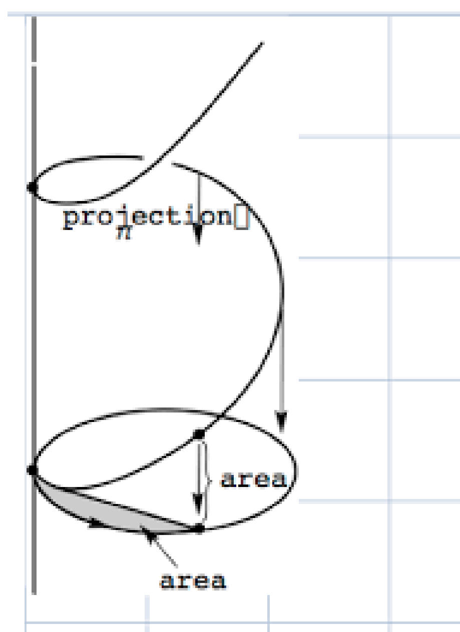



Figure 33: A carnot curve, image by [14]

Now for a domain $\Omega$ we have $\text{Area}(\Omega) = \int_\Omega dxdy = \int_{\partial\Omega} \frac{1}{2} xdy - ydx$, hence for any curve $(x(t), y(t)$ through the origin in $\mathbb{R}^2$, the encoded curved $(x(t), y(t), z(t)) \in \mathbb{R}^3$ where $z$ is the area of the segment bounded by the curve and the straight line between its endpoints is a Carnot curve. Define the length of a Carnot curve to be the length of its Euclidean projection (splitting along a suitable point if necessary). The dual problem of the isoperimetric problem in the plane is to find a curve of shortest length with given area, thus we are asking for the shortest Carnot curve that ends at a height given by the given area. But we know the solution to this are circles, in other words the Carnot curves are exactly lifts of circles (to the height which is the area between them and the line segment connected $(0, 0)$ to $(x, y)$ - thus we get a helical structure. This turns the Heisenberg group into a metric space.

As before we can consider the action of the Heisenberg group on itself, really we may declare these as isometries. The full isometry group is the semi-direct product of this with the point stabiliser O(2). These give us the geodesics are vertical lines and *Legendrian curves* - those curves in $\mathbb{R}^3$ with tangent vectors lying in the plane field given by associating $(x, y, z)$ with $\langle(1, 0, 0), (0, 1, x)\rangle$ (travelling along this you can easily see the helical structure).

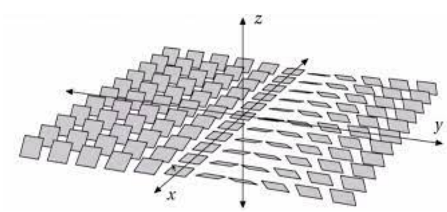



Figure 34: The contact structure for the Heisenberg group, the Legendrian curves are the paths with tangents parallel to this

The final geometry in this part of the family is the wonderfully named geometry of the universal cover of $\text{SL}(2, \mathbb{R})$. This is sometimes known as twisted $\mathbb{H}^2 \times \mathbb{R}$, with the same point stabilisers as twisted Euclidean. The best way to understand this (which in fact classifies all the objects of this geometry) is to look at Seifert fiber spaces. These are essentially formed by taking a solid torus, cutting it (at an obvious meridian) and gluing the ends of the solid cylinder with a rational twist. The fibers are still circles (lines going vertically through the cylinder) but say in general will not loop around the same number of times, the one through the center will only loop around once.

## 2.5 A classification of 3-manifolds

As we have seen, three dimensions is a lot cooler than two, and whilst there are many similarities it is not easy to classify them. In fact we saw earlier that we required a model geometry to have a compact manifold modelled on

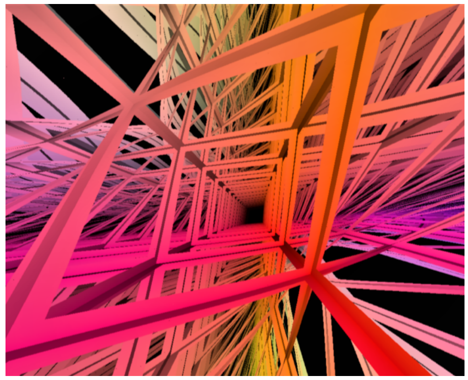


Figure 35: Geometry of $\mathrm{SL}(2,\mathbb{R})$, image by [13]

it, without this there are uncountably many geometries. The first classification comes from looking at point stabilisers, which is what we focused on before, I will write out a formal argument for what these can be since I originally found it confusing. The idea is our usual obvious isomorphism type theorems work since we have nice enough spaces, followed by classic connnectedness arguments.

For a model geometry $(G, X)$, we consider the identity component $G'$, the action of this is transitive, indeed fix a $x \in X$ and consider the map $G \to X, g \mapsto gx$. Then by transitivity of $G$ this descends to a well defined continuous bijection $G/G_x$ (where $G_x$ is the stabiliser) for which the domain contains the image of $G'$ as an open set by definition of the quotient map (and that components are clopen). This is a continuous bijection, but $G$ is a Lie group so it is in fact a homeomorphism, see proposition 1.2 in [10] - this is not obvious. Thus the orbit of $x$ under $G'$, which is just the image of $G'/G_x$ under the above map is also clopen, so we are done by connectedness of $X$. Now $\coprod_x G'_x/(G'_x)_0$ forms a covering space of $X$ (where $(G'_x)_0$ is the identity component in the stabiliser), which can be justified by the definitions noting that these stabilisers vary continuously. But $X$ is simply connected so any covering is trivial, hence $G'_x$ is connected and compact. Now for any point we can consider a small ball around it - this is the idea that on a very small scale our geometries are Euclidean, hence we are looking for connected closed subgroups of SO(3), but since this is a Lie group any such subgroup must also be a Lie group. Suppose we have such a non-trivial subgroup H of SO(3), by parameterising (continuously) its elements by an angle of rotation, that it contains the identity and is connected, it must have some rotation by an irrational amount about some axis (e.g. must contain some small enough interval), but the subgroup generated by this is dense so by closure we get it contains all rotations about this axis. So if H is one-dimensional, it must be a copy of SO(2). Now suppose H is at least two-dimensional, so it must a rotation about some other axis. Note that a reflection along the x-axis followed by one along the y-axis is a reflection along the z-axis, but rotations are compositions of such, so if we have two orthogonal eigenvectors we must have that H is all of SO(3). The idea now is that no matter how close one of the eigenvectors is to the other, we can compose rotations of half a turn about each axis to get further and further away, till the angle between our final vector and original is a little greater than $\frac{\pi}{2}$, so the inner product is negative. By connectedness at some point its zero, so we are done by the orthogonal case above.

Now for the three dimensional point stabilisers, we get things isotropy and homogeneity which gives us constant sectional curvature, forcing a Euclidean, spherical or hyperbolic structure. For one-dimension, we put a foliation given by the $G'$ invariant vector field produced from the axis of rotation associated to the one dimensional stabilisers SO(2). The flow commutes with the group action and maps the tangent plane isometrically along a leaf, and so the leaf must be the image of $S^1$ or $\mathbb{R}$ since it does not accumulate. Quotienting out by this leaf we get a two dimensional simply connected space $Y$ which inherits a metric from $X$. The plane field orthogonal to the leaf has constant curvature (by transitive action). The zero curvature case gives us $Y \times E$, where $Y$ is $E^2, S^2$ or $\mathbb{H}^2$ - only two are new. The case where the curvature is non-zero we get a contact structure as discussed before, which splits into cases of $Y$ as above, if $Y = S^2$ we get back $S^3$ and the Hopf fibration. If $Y = \mathbb{H}^2$ we get nilgeoemtry and otherwise the geometry of the universal cover of $\mathrm{SL}(2, \mathbb{R})$. The one-dimensional stabiliser is a little more involved from which the only new geometry is solgeometry, this can be understood a little by considering the following form taken as a Lie group formed by matrices $\left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$. Geodesics connecting two points are not unique though and it is the least symmetric of all the geometries.

Looking at these foliations actually almost gives us another characterisation. I mentioned Seifert fiber spaces before and it's only solgeometry and $\mathbb{H}^3$ that have some issues with it. Moreover $S^3$ and $E^3$ are the ones out of the remaining for which there is not really a canonical foliation, but the ideas still work well enough.

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