The Local Langlands Correspondence for GLn

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1 Introduction

The Langland's programme aims at making strong connections between algebraic or geometric objects and analytic objects, an instance of this links geometric Galois representations of the absolute Galois group, which is a huge object in number theory, to analytically originated automorphic representations of reductive algebraic groups over the adeles living in representation theory. A motivating correspondence is Artin reciprocity, a generalisation of quadratic reciprocity, which shows that understanding Galois groups of abelian extensions of number fields gives genuine arithmetic data of the field. We can also go the other way, the correspondence associates many numbers attached to the representations; it is harder to show properties of L-functions such as analytic continuation on the Galois side but is proven and used extensively in analytic number theory, for example in Dirichlet's theorem on arithmetic progressions. Artin reciprocity is at the heart of class field theory which is incredibly rich. Ultimately, the absolute Galois group encodes a huge amount number theoretic data that class field theory can't access; to retrieve much of it we would need a non-abelian generalisation of Artin reciprocity - this is what the Langlands correspondence builds upon.

I aim to define the objects of the local Langlands correspondence, this has both a precise statement and has been proven, but at times, where relevant, global connections will be made. An important theme in local fields is to filter the units using unit groups in which see how close a unit is to 1; these are a lot harder to understand and the filtration makes it much easier to then use analytic tools such as a p-adic logarithm. Going deeper in the filtration makes it harder, but due to a result of Grothendieck naturally arising representations that come into play in the Galois side almost trivialise the action beyond the inertia. These analogous structures on the two sides are starkly connected by local Artin reciprocity in the abelian case with image the (abelianisation of the absolute) Weil group, so the reciprocity map is a topological isomorphism and characterised by certain properties.

The higher dimensional analogue is to consider certain n-dimensional representations of the Weil group and match them up to certain representations of a system of reductive algebraic groups, I will look at the case of GL_n over a p-adic local field. The local Langlands correspondence doesn't give something as strong an 'isomorphism' as in the abelian case but a lot of nice theory is developed to define it and match certain local factors. There are many key features that arise from smaller cases, for example it heavily relies on simple generalisations of the two filtrations, which is why the intuition above is important in order to have a canonical matching of representations and their properties. It is also crucial that the Weil group is considered here as the inertia being open means it admits many more representations than the absolute Galois group where open subgroups are very large having finite index which can be used to show that continuous representations in fact have finite image. The proof by Harris and

Taylor uses Shimura varieties which lie in between the two sides providing a concrete link and gives a geometric flavour in the theory of both sides, however I will not discuss the proof.

The first section is on certain Weil-Deligne representations which can be viewed more naturally as semisimple representations of the larger Weil-Deligne group. The source is geometric, coming from elliptic curves and more generally algebraic varieties such as abelian varieties, the representation theory thus admits a series of beautiful geometric reductions to essentially combinatorial building blocks. The one-dimensional case is particularly rich in theory when attaching various numbers to the representations, which exploits the theory of Fourier analysis accessible on locally compact topological groups, developed by Tate in this case. Whilst this section is quite geometrically flavoured Galois theory, the second section is analytically flavoured representation theory; many objects and intuition and comes from analysis as for example we will see with Hecke algebras. However, there are still many geometric motivations in the representation theory of GL_n. A lot of the techniques from the case of finite fields or $n = 2$ can be generalised, in the latter case the correspondence is quite explicit. What makes the gears turn in the geometric reductions of this side is the Bernstein-Zelevinsky classification which is again quite combinatorial in essence - as expected if one believes the correspondence. Since there is a lot of theory on both sides, I will place a particular emphasis on the Galois side and attempt to mirror the development on the side of representation theory.

Finally, I will state the correspondence and briefly explain its relation with class field theory, the presentation of this is from Harris and Taylor which is serves as a template for other flavours of Langlands correspondences. What I really love about this theory is the bridge of two seemingly unrelated continents, each in itself a synthesis of different branches of mathematics with their own surprising and deep number theoretic and geometric relations. There are so many exciting directions and tunnels that emerge, many of which are active areas of research.

Notation

This is to avoid repetition in definitions or results. Unless stated otherwise:

- $K/\mathbb{Q}_p$ a p-adic local field for $p \in \mathbb{Z}$ fixed prime, absolute value $|\cdot| = |\cdot|_K$, residue field $k/\mathbb{F}_p$ size $q = p^a$, valuation ring $\mathcal{O}_K$, uniformiser $\pi = \pi_K$ with $|\pi|_K = q^{-1}$
- Maximal unramified extension K^{ur}/K and maximal tamely ramified extension $K^{\mathrm{tr}}/K^{\mathrm{ur}}$ inside fixed separable closure $\bar{K}/K$
- Absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$, Weil group W_K, inertia $I_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{ur}})$, wild inertia $P_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{tr}})$ and $\Phi \in W_K$ lift of the (arithmetic) Frobenius $\mathrm{Frob}_k \in \mathrm{Gal}(\bar{k}/k)$ corresponding to $1 \in \mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$.

2 Galois Theoretic Side

The Weil group occurs naturally from local reciprocity and a representation will be defined so that it also respects the topologies, but it turns out it helps package certain geometric representations with incompatible topologies into families. Having more open subgroups makes its representation theory a lot more interesting, but it also means we have a lot of representations that are infinite even on the inertia which we intuited early to be hard to fully understand. A natural source of representations are ℓ-adic ones and in the proof of Grothendiecks ℓ-adic monodromy theorem we see that P_K acts with finite image, so in this sense these representations are not too complicated. The flow of the first half of this section starts by understanding where Weil-Deligne representations come from and that the irreducible ones have a pretty simple structure; they look like twists by the absolute value of pretty simple representations. They can be realised as representations of the Weil-Deligne group and we look at the representation theory of this group, the semisimple ones turn out to be combinatorial modifications of the irreducible representations called special representations which form the building blocks. The main reference for this section is Tate's Number theoretic background [Tat93].

2.1 ℓ-adic Representations

To motivate the definition of a Weil-Deligne representation, we first look at ℓ-adic representations, a special source of examples come from the Tate module of an abelian variety. There is a huge amount of interesting theory but I will only briefly touch upon the certain aspects such as ℓ-independence as in Fontaine and Ouyang [FO].

2.1.1 Geometric Source

The first thing we want in a representation of a topological group such as G_K is continuity, but a complex representation is forced to factor through a finite quotient by continuity and a no small subgroups argument. So we might put some continuity conditions relating to $\mathbb{Q}_\ell$ (for some prime $\ell \in \mathbb{Z}$) after simultaneously embedding all these into $\mathbb{C}$, but if we range over different ℓ then there is little reason to expect that the representation is continuous for all ℓ as $\mathbb{Q}_\ell$, for various primes, behave very differently in a topological sense, so we focus in on one prime only.

Definition. *A continuous representation $\rho : G_{L/K} \to GL_n(\mathbb{Q}_\ell)$ is an ℓ-adic representation of $G_{L/K}$. If $L = \bar{K}$, say ρ is an ℓ-adic Galois representation.*

By the embedding of $\mathbb{Q} \hookrightarrow \mathbb{Q}_\ell$ and injection $G_{\mathbb{Q}_\ell} \to G_{\mathbb{Q}}$ using density and continuity, we can take an ℓ-adic representation of the absolute Galois group and restrict to the local case. It is enough to define a representation on a lattice; for a continuous linear action $\rho : G \to \mathrm{GL}_n(\mathbb{Z}_\ell)$, we obtain an ℓ-adic representation $\rho : G \to \mathrm{GL}_n(\mathbb{Q}_\ell)$. From now we take $\ell \neq p$.

Definition. *For an ℓ-adic representation $\rho : G_K \to GL_n(\mathbb{Q}_\ell)$, say ρ*

(i) has good reduction (or is unramified) if I_K acts trivially, equivalently $\rho(I_K) = 1$. It has potentially good reduction if it has good reduction after passing to a finite extension of K, equivalently $\rho(I_K)$ is finite,

(ii) is semistable if its semisimplification has good reduction, equivalently I_K acts unipotently. It is potentially semistable if its semisimplification has potentially good reduction.

The representations of the first kind are relatively easy to understand, indeed factoring through $G_K/I_K \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$ gives a nice explicit group to work with. The semistable cases mean we instead almost factor through G_K/P_K, but this is given by the semidirect product of $\hat{\mathbb{Z}}$ and its puncture at $\mathbb{Z}_l$ so these representations only have a finite image on the hard bit, this is important in the global case of trying to work out which Galois representations come from geometry as part of a conjecture by Fontaine and Mazur. If A is an abelian variety of dimension g over K of characteristic prime to ℓ, then

$$A(\bar{K})[\ell^n] \cong (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$$

with natural action of G_K on these $\bar{K}$. With appropriate compatible maps, the Tate module of A is

$$T_\ell(A) = \varprojlim A(\bar{K})[\ell^n] \cong \mathbb{Z}_\ell^{2g}$$

packaging its torsion and giving a representation $G_K \to \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$. For an elliptic curve E/K if $T_\ell(E)$ is unramified then the characteristic polynomial of the image of the Frobenius is $T^2 - aT + q$ where $a = \mathrm{tr}(\Phi) = q + 1 - |\tilde{E}(k)|$ - giving another instance of ℓ-independence.

Another source of ℓ-representations come from the étale cohomology of an algebraic variety over a local field K. In the case of coefficients in $\mathbb{Q}_\ell$ and an abelian variety over K of residue characteristic for which ℓ is prime to, the first étale cohomology is the dual of (the scalar extension of) the Tate module. This is the essential source of ℓ-adic representations, and there is a sense in which all these geometric representations come from subquotients of the étale cohomology (described in [FO]). However, there are still pathological constructions, for example restricting $G_K \to G_k \cong \hat{Z} \to \mathbb{Z}_\ell$ and we can be quite brutal with the image of 1.

2.1.2 ℓ-adic Monodromy Theorem

We now show that these representations from geometry are very nice and in fact are potentially semistable. The crux of difficulty of understanding W_K comes from P_K, assuming this plays no role we turn to the linearisation of automorphisms.

Definition. *The composition $t : I_K \to I_K/P_K \to \prod_{\ell \neq p} \mathbb{Z}_\ell$ is the tame character with $\sigma \mapsto t(\sigma) = (t(\sigma)_\ell)$ where $\sigma(\pi^{1/\ell^n}) = \zeta_{\ell^n}^{t(\sigma)_\ell} \pi^{1/\ell^n}$*
The ℓ-part is the composition $t_\ell : I_K \xrightarrow{t} \prod_{\ell \neq p} \mathbb{Z}_\ell \to \mathbb{Z}_\ell$

So we have an easy arithmetic way of talking about its eigenvalues. We also have the unramified character of W_K as the composition

$$\omega : W_K \xrightarrow{\text{res}} W_K^{\text{ab}} \xrightarrow{\text{Art}^{-1}} K^\times \xrightarrow{|\cdot|} \mathbb{C}^\times$$

Where Art is the local Artin isomorphism. Thus, $\omega(I_K) = 1$, $\omega(\Phi) = q^{-1}$ so its image is in $\mathbb{Q}^\times$. These two are intimately related:

Proposition 2.1. *$\forall x \in W_K$, $\sigma \in I_K$, we have $t(x\sigma x^{-1}) = \omega(x)t(\sigma)$*

Proof. This follows immediately from reducing to checking for $x = \Phi$ (true for I_K as target is abelian) and the essence of Kummer theory as we only have to check $\Phi\sigma^q = \sigma\Phi$ on $K^{\text{tr}} = K^{\text{ur}}(\{\sqrt[n]{\pi}|(n,p) = 1\})$ as we work modulo P_K. $\square$

Analogously, we have $\forall g \in G_K$, $\sigma \in I_K$,

$$t_\ell(g\sigma g^{-1}) = \chi_\ell(g)t_\ell(\sigma)$$

where χ_ℓ is the (unique) ℓ-adic cyclotomic character. Here again we can verify on a Frobenius and use continuity, but it is important that we work with χ_ℓ as earlier we took an embedding of the rationals into the ℓ-adic rationals.

Theorem 2.2. (*Grothendieck's ℓ-adic Monodromy*) *ρ an n-dimensional ℓ-adic representation of W_K, then*

- *(i) $\exists$ open subgroup $H \leq I_K$ for which $\rho(x)$ is unipotent $\forall x \in H$.*
- *(ii) $\exists!$ nilpotent endomorphism N of $GL_n(\mathbb{Q}_\ell)$ such that $\forall x$ in a sufficiently small open subgroup of I_K we have $\rho(x) = \exp(t_\ell(x)N)$*

Proof. Note since $\text{GL}_n(\mathbb{Z}_\ell)$ is a maximal compact subgroup and I_K is compact, we may conjugate ρ so that the image of I_K is in $\text{GL}_n(\mathbb{Z}_\ell)$. The first step is important; P_K acts with finite image. This comes from considering the separated family of open subgroups $U_i = 1 + \ell^i\text{GL}_n(\mathbb{Z}_\ell)$ and the key fact that P_K is profinite with pro-order prime to ℓ, which means the preimage of U_1 in P_K is open and contained in $\ker(\rho) \cap P_K$ by successively considering its image in U_i/U_{i+1} which is a p-group.

This means we should be able to understand ρ using the linear theory we have above, or at least after passing to some (large) open subgroup of I_K as P_K doesn't quite act trivially. In fact something quite strong happens, since $\ker\, t_\ell \cong \prod_{q\neq\ell,p}\mathbb{Z}_q$ has pro-order prime to ℓ, the image of this is also finite. So their kernels must share a large open which comes from some $H \leq I_K$, so ρ factors through t_ℓ on H, and shrinking H so its normal means $\rho(x)$ is conjugate to $\rho(x)^q$ for $x \in H$ by 2.1. This puts a considerable restriction on its eigenvalues; they are roots of unity, but taking the ℓ-adic logarithm to eliminate torsion (further shrinking H if necessary) forces all the eigenvalues to be 1, hence $\rho(x)$ is unipotent.

For (ii), we define $N = \frac{\log\rho(y)}{t_\ell(y)}$ for some $y \in H$. Factoring $\exp(t_\ell(x)N)$ and $\rho(x)$ via t_ℓ shows they are equal on some small neighbourhood, noting they agree at $t_\ell(y)$ and so the closure of the subgroup generated by it which is open in $\mathbb{Z}_\ell$. $\square$

Thus, the representations from geometric sources we saw before are all potentially semistable. With a little more work, 2.2 holds analogously for ℓ-adic representations of G_K. There is an analogue of the monodromy theorem for $\ell = p$, here the topologies interact meaningfully and is studied in p-adic Hodge theory! There is a lot more that can be discussed here, but I will save it for after defining a Weil-Deligne representation.

2.2 Weil-Deligne Representations

The main motivation is 2.2 which effectively says there is a better formal way of defining ℓ-adic representations. From there we have a nice way to classify all semisimple representations, though this definition effectively forgets the nilpotent operator. The main ideas of the section are that 2.1 gives a way to propagate a representation; multiplying by ω, and importantly that we can always pass to a small enough subgroup of I_K to trivialise the action. I will mostly follow Deligne [Del73a] and [Del73b].

2.2.1 Representations of W_K

The statement of 2.2 (ii) says we can modify an ℓ-adic representation so that all the difficulty is formalised and wrapped up in a nilpotent operator so we only really care about representations that are finite on the inertia, the nilpotent operator will be remembered later. First we classify representations of W_K.

Definition. *A representation of W_K on a vector space V is a continuous homomorphism $\rho : W_K \to GL(V)$.*
Since W_K is dense in G_K, $Rep(G_K)$ is a subset of $Rep(W_K)$: any such representation is of Galois-type.
Say ρ is unramified if the inertia acts trivially, and ramified otherwise.

From now, assume any representation of W_K is over a $\mathbb{C}$-vector space, the reason for doing this is if the inertia has finite image then the representation is continuous with respect to the discrete topology, so we may as well take a $\mathbb{C}$-vector space (embedding any suitable field into $\mathbb{C}$). The 'no small subgroups' argument shows ρ is trivial on some finite index subgroup of I_K by continuity, the converse is also true - being trivial on a finite index open of the inertia means it is continuous.

We already have ω is a one-dimensional representation (or quasi-character) that is unramified. Conversely, any one-dimensional representation trivial on inertia is of the form $\omega^s := (\omega(\cdot))^s$, some $s \in \mathbb{C}$ - these are not of Galois-type. The Galois-type representations on W_K are very restricted, these are exactly the representations for which W_K has finite image (no small subgroups argument). However, the two above examples do give the whole story:

Proposition 2.3. *Every irreducible representation ρ of W_K is of the form $\rho = \rho' \otimes \omega^s$ for some Galois-type ρ', $s \in \mathbb{C}$.*

Proof. This proof is adapted from Deligne [Del73a]. Using the 'no small subgroups' argument again; ρ is trivial on some finite index subgroup J of I_K, take the largest such. The idea is to define the type of a representation, and break up $\mathrm{Rep}(W_K)$ accordingly, which, for a given type, can then be realised as a twist of a Galois-type representation with a quasi-character of the same type.

Let n be the smallest positive integer such that Φ^n acts trivially by conjugation on I_K/J. If Φ^n has eigenvalue λ, take an eigenvector v of Φ with n^{th}-power eigenvalue λ, then the subspace generated by $\{\sigma(v) \mid \sigma \in I_K/J\}$ is invariant, so must be the whole space by irreducibility. We define the type of ρ to be λ.

For example, running through the argument with ω, we see it has type q^{-1} and so ρ will have the same type as ω^s for some $s \in \mathbb{C}$. Also, the type 1 irreducible representations are exactly the irreducible Galois representations, since ρ is trivial on J and the subgroup generated by a power of Φ, so the kernel has finite index. Thus, twisting ρ by ω^{-s}, we get a type 1 irreducible representation ρ'. $\square$

Corollary 2.4. *There is an equivalence of categories*

$$Rep(G_K) \to Rep_\lambda(W_K)$$

by twisting with ω^s of type λ. Here $Rep(W_K) = \bigoplus_\lambda Rep_\lambda(W_K)$ is the direct sum of abelian sub-categories of representations of type λ.

We can build more representations in the usual way, say by taking direct sums, tensor products and duals. For a finite extension L/K we can also restrict and induct representations, satisfying the usual Frobenius reciprocity. Given an ℓ-adic representation of G_K, it may be infinite on the inertia, and so (embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$) will not give a (complex) representation of W_K. The monodromy theorem motivates the following:

Definition 2.5. *The Weil-Deligne group W'_K, is the semidirect product $W_K \ltimes \mathbb{C}$, with action $gxg^{-1} = \omega(g)x$, equipped with the product topology. A representation on V is a continuous homomorphism $W'_K \to GL_n(\mathbb{C})$, for which the restriction to $\mathbb{C}$ is analytic.*
Since finite dimensional complex representations of the additive group $\mathbb{C}$ correspond to nilpotent operators, we can realise such a semisimple representation as a pair $\rho' = (\rho, N)$ where

(i) $\rho' : W_K \to GL_n(\mathbb{C})$ *is a representation continuous with respect to the discrete topology.*

(ii) N *is a nilpotent operator such that* $\rho(g)N\rho(g)^{-1} = \omega(g)N$.

Say ρ' is a Weil-Deligne representation of W_K.

So a subspace invariant under $W_K \ltimes \mathbb{C}$ is a subspace invariant under both W_K and N. We thus have a notion of irreducibility, semisimplicity and indecomposability for pairs (ρ, N).

The first tells us the representation is not influenced by the underlying topology and really has been eliminated, the second condition encodes the complexity of I_K in the operator. To avoid confusion in notation and for practicality, I will generally drop the use of the Weil-Deligne group and work with pairs (ρ, N), but most of the notions described, such as semisimplicity, are naturally defined on W'_K and reinterpreted in terms of the pair.

2.2.2 Structure of Weil-Deligne Representations

To understand how to take sums, tensors and duals of such pairs, its easiest to work them out for ℓ-adic representations of W_K and see what they mean for the Weil-Deligne representation, it turns out we actually hit all of them. Denote by $\mathrm{WD}(W_K)$ the category of Weil-Deligne representations, where a morphism of two such is a linear map of the underlying vector spaces, commuting with the corresponding representations of W_K and nilpotent operators. For the following theorem, we consider only those Weil-Deligne representations on vector spaces with coefficients in an ℓ-adic local field for $\ell \neq p$.

Theorem 2.6. *There is an equivalence of categories*

$$\left\{\begin{matrix} \ell\text{-adic representations} \\ \text{of } W_K \end{matrix}\right\} \xrightarrow{\rho \mapsto (\rho', N)} \left\{\begin{matrix} \text{Weil-Deligne representations} \\ \text{of } W_K \end{matrix}\right\}$$

where $\forall x \in I_K$, $\rho'(\Phi^n x) = \rho(\Phi^n)\rho(x)\exp(-t_\ell(x)N)$, *and* N *is the monodromy operator of* ρ *from 2.2.*

Proof. This is really a rephrasing of the monodromy theorem, with a few more checks, though each one follows a similar flavour.

First we see (ρ, N) defines a Weil-Deligne representation by 2.2. Using 2.1 we get that ρ' is a group homomorphism and the second condition in the definition of a Weil-Deligne representation, noting that $\rho N \rho^{-1} = \omega N$ holds for ℓ-adic representations. The first condition holds as ρ' is trivial on some finite index subgroup of I_K and surjectivity follows by retrieving ρ from ρ' via its definition. Faithfulness follows from the fact that a morphism commuting with the ρ's commutes with the exponential term of the monodromy operators, that is, for a morphism α for two ℓ-adic representations ρ_1, ρ_2 with monodromy N_1, N_2 we have $\alpha \circ \rho_1 = \rho_2 \circ \alpha$ and $\alpha N_1 = N_2 \alpha$ by writing the monodromy in terms of logarithms. ☐

Corollary 2.7.

(i) $(\rho_1, N_1) \oplus (\rho_2, N_2) = (\rho_1 \oplus \rho_2, N_1 \oplus N_2)$

(ii) $(\rho_1, N_1) \otimes (\rho_2, N_2) = (\rho_1 \otimes \rho_2, N_1 \otimes 1 + 1 \otimes N_2)$

(iii) $(\rho, N)^\vee = (\rho^\vee, -N^\vee)$

(iv) *Hom*$((\rho_1, N_1), (\rho_2, N_2)) = ($*Hom*$(\rho_1, \rho_2), N)$ *where* $(N\theta)(v) = N_2(\theta v) - \theta(N_1 v)$

Proof. These follow by tracing what happens under the equivalence of 2.5: (i) is immediate, (ii) follows from $\log(\rho_1(y) \otimes \rho_2(y)) = \log((\rho_1(y) \otimes 1)(1 \otimes \rho_2(y)) = (\log(\rho_1(y))) \otimes 1 + 1 \otimes (\log(\rho_2(y)))$ by expanding $\log(x \otimes 1)$ as a series in $x \otimes 1 - 1 \otimes 1$, finally (iii) is a special case of (iv) which follows by a similar computation using properties of the logarithm. $\square$

There is an analogue of 2.5 for ℓ-adic representations of G_K, where the right side is replaced with bounded Weil-Deligne representations - the eigenvalues of the image of any element are in $\mathbb{Z}_\ell^\times$ (enough to check on Φ). In fact, this can be proven using 2.5 by noting that a representation of W_K extends to G_K iff the map $1 \mapsto \rho(\Phi)$ extends to $\hat{\mathbb{Z}}$ (by factorising $g = \Phi^a x, a \in \hat{Z}, x \in I_K$), which, by linear algebra, is equivalent to boundedness. The equivalence of categories is then identified using 2.5 where the notions of irreducibility and semisimplicity pass over in this case.

Since a subspace of a Weil-Deligne representation is W_K'-invariant if it is invariant under W_K and N, we have $\rho = (\rho', N) \in \mathrm{WD}(W_K)$ is irreducible iff ρ' is irreducible and $N = 0$ (as the kernel of N is W_K-invariant). This is quite restrictive, so we drop the condition of invariance under the nilpotent operator, which ends up corresponding to the notion of indecomposability on the Weil-Deligne group. From now, we identify the irreducible representations of W_K with the Weil-Deligne representations $(\rho, 0)$ with ρ irreducible.

Definition 2.8. *A Weil-Deligne representation (ρ, N) is Frobenius-semisimple (or F-semisimple) if ρ is semisimple.*

Given any $(\rho, N) \in \mathrm{WD}(W_K)$ on V, its semisimplification is the induced representation (ρ^{ss}, N) on $\bigoplus V_i/V_{i-1}$ where $V = V_n \supset ... \supset V_0 = 0$ with each V_i/V_{i-1} irreducible. We now justify its name; it is enough to check semisimplicity on a choice of Frobenius.

Fact. *If V is a complex vector space, then an endomorphism T of V is semisimple iff T is diagonalisable.*

So T is semisimple iff some power of T is semisimple. It also shows a commuting family of semisimple endomorphisms of V is simultaneously semisimple; any subspace invariant under the whole family has a complement invariant under the family.

Lemma 2.9. *A Weil-Deligne representation (ρ, N) is semisimple iff $\rho(\Phi)$ is semisimple.*

Proof. The proof mostly reduces to linear algebra; the forward direction holds from the fact above. It also shows that it is enough to prove: $\rho(\Phi)$ is semisimple implies $\rho(x)$ is semisimple $\forall x \in W_K$. The key idea is to use finiteness of the image of inertia, as this enables to pick a finite set of representatives for which any large enough power of an element acts trivially by conjugation on (and hence commutes), and that $\rho(\Phi)$ has finite index in $\rho(W_K)$. The diagonalisability then follows by the fact again. $\square$

One can use the fact and that ρ is semisimple iff its restriction to a finite index subgroup is semisimple to give a faster proof - here this would be the subgroup generated by Φ in $W_K/\mathrm{ker}\rho$.

2.2.3 Special Representations

We now look at a key example producing F-semisimple Weil-Deligne representations from an irreducible representation of W_K. These are defined by looking at a tuple of the unramified character ω, where successive coordinates are related by a twist of ω - so we can think of it as an interval. The reason for looking at this representation is that conjugation by ρ on N acts by multiplication by ω, so on invariant subspaces we naturally get a propagation by successive powers of ω but abruptly stops due to nilpotency of N - it turns out this pretty much gives the whole picture. This is also motivated by looking at representations of elliptic curves, in particular one of split multiplicative reduction which has local polynomial $1-T$ thus giving a one-dimensional invariant subspace, and the after some computations one retrieves the special representation (of dimension 2) as defined below. There is a very clear analogous structure on the representation theoretic side which the special representation reflects, we will see this in the Bernstein-Zelevinsky classification.

Define the special representation (sp_m, M) as follows. Take the standard complex basis $e_1, ..., e_m$, then

(i) $sp_m = \bigoplus_{i=0}^{m-1} \omega^i : W_K \to \mathrm{GL}_m(\mathbb{C})$

(ii) $Me_m = 0$, $Me_i = e_{i+1}$ for $i \leq m-1$

An easy check shows $sp_m(x)M = \omega M sp_m(x)$, so $\mathrm{ker}M = \mathbb{C}e_m$ is an invariant subspace of dimension 1, which shows it is an indecomposable Weil-Deligne representation (as it can't split) which is reducible for $m \geq 2$. For an irreducible representation $\rho : W_K \to \mathrm{GL}_n(\mathbb{C})$, define

$$sp_m(\rho) = \rho \otimes sp_m$$

to be the nm-dimensional Weil-Deligne representation, with operator as in 2.7.

Using this, we can show that the irreducible continuous representations of W_K build to give F-semisimple Weil-Deligne representations. Denote WD_n^{Fss} to be the n-dimensional F-semisimple representations in $\mathrm{WD}(W_K)$, and $\mathrm{WD}_n^{\mathrm{indec}}$ to be the representations in WD_n^{Fss} that are indecomposable as representations of the Weil-Deligne group.

The picture is that for $(\rho, N) \in \mathrm{WD}_n^{Fss}$ acting on V, (ρ, N) gives a belt - much like in Lie algebras - a sequence of ρ-subrepresentations $V_1, ..., V_{m+1} = 0$ such that $NV_i \subseteq V_{i+1}$. By definition, $\bigoplus V_i$ is a subrepresentation of (ρ, N) and $\rho(x)N = \omega N \rho(x)$ shows the restriction of ρ to the image of N in V_{i+1} is isomorphic to the twist by ω of the action of ρ on V_i (choose a basis of V_i

inductively so its image under N extends to a basis of V_{i+1}). For example,

$$sp_m(\rho) = \rho \oplus \cdots \oplus \rho(m-1)$$
$$N = 1 \otimes M$$

where $\rho(i) = \rho \otimes \omega^i$ gives this picture with each block given by $V_i = \mathbb{C}^n \otimes e_i$ - so N moves these blocks killing the last one. For $(sp_m(\rho), N')$ where $N' = 1 \otimes N$, we have for W_K-invariant subspaces $U \oplus V$, that $(U \cap \ker N') \oplus (V \cap \ker N')$ is an invariant subspace. For ρ irreducible note that $\rho \otimes \omega^i$ is irreducible for any i, so $\ker(1 \otimes N) = \mathbb{C}^n \otimes e_m$ is irreducible so can't split into a direct sum of invariant subspaces, so these blocks are quite rigid and unbreakable, the only movement is horizontally by N. This shows the first half of:

Lemma 2.10. *The special representation is indecomposable. Conversely, any indecomposable Weil-Deligne representation is special. So we have a bijection*

$$\begin{aligned} \{(m,\rho) | m \geq 1, \rho \in Rep_n(W_K) \text{ irreducible}\} &\longleftrightarrow WD^{indec}_{nm} \\ (m,\rho) &\longmapsto sp_m(\rho) \end{aligned}$$

The idea for the converse is to take an isotypic component and choose an irreducible representative, the block then encodes the copies of this inside the isotypic component reducing to linear algebra, details are in [Del73a]. A more concrete understanding as to there is a flavour similar to Lie algebras actually comes from an identification of representations of W_k' and those of $W_K \times SL_2(\mathbb{C})$, the representation theory of the latter gives representations of its Lie algebra and can be seen to produce the special representation. We can now build all the F-semisimple Weil-Deligne representations. A virtual representation is a formal sum of representations with integral coefficients, denote $R(G)$ the group of virtual representations of G where

$$[V] = [U] + [W] \text{ if } V \cong U \oplus W$$

This gives us a way to subtract representations, in the way $\oplus$ adds them, but doesn't necessarily return a representation. Deligne [Del73b] shows

Lemma 2.11. *A representation ρ of W_K can be represented as*

$$[\rho] = dim\ \rho[1_K] + \sum_{(L,\chi,\chi')} c_{L,\chi,\chi'}\ Ind^L_K(\chi - \chi')$$

where the sum runs over all finite L/K and pairs of quasi-characters of W_L and the integral coefficient is 0 for all but finitely many triples.

This will be especially important for L and ϵ-factors giving them an inductive property. Generally this is used as an induction tool, for example to show

Corollary 2.12. *There is a bijection*

$$\begin{aligned} \{(n_i,\rho_i)_i | n_i \geq 1, \sum_i n_i = n, \rho_i \in WD^{indec}_{n_i}\} &\longleftrightarrow WD^{Fss}_n \\ (n_i,\rho_i)_i &\longmapsto \bigoplus_i \rho_i \end{aligned}$$

The formal statement 2.2 (ii) tells us an ℓ-adic representations can be understood in terms of the ℓ-part of the tame character and a nilpotent operator. Since the tame character stitches all the ℓ-parts together, $\rho(x)\exp(-t_\ell(x)N)$ has a chance to show ℓ-independence. The definition of a Weil-Deligne representation associated to an ℓ-adic representation does not involve the topology of $\mathbb{Q}_\ell$; it untangles any (weak) interactions of the topologies. In fact, it does so very well, giving rise to a "compatible family" of ℓ-adic representations.

Theorem 2.13. *[FO] For an abelian variety A/K and Tate modules $T_\ell(A), T_{\ell'}(A)$ where ℓ, ℓ' are primes different to p, the associated Weil-Deligne representations are isomorphic.*

2.3 L and ϵ-factors

The Riemann-zeta function has very interesting properties; satsifying a functional equation, admitting a meromorphic continuation and having an Euler product. A Dirichlet series having these properties is an L-function, and using this we can prove some very interesting results, such as (generalisations of) Dirichlet's theorem on arithmetic progressions. To a number field L, we can associate a generalisation of the zeta function; the Dedekind zeta function

$$\zeta_L(s) = \sum_{I \leq O_L} N(I)^{-s}$$

a similar technique, by considering the theta function and using the Poisson summation formula, shows this also has the desirable properties by considering a completed zeta function, notably a gamma function crops up. The completed zeta function, expressed as an Euler product (with a factor at infinity)

$$Z_L(s) = |d_L|^{\frac{s}{2}} \cdot (\Gamma_L(\frac{s}{2})^{r_1} \cdot 2^{-(s-1)r_2}(\Gamma_L(2)^{r_2} \cdot zeta_L(s)$$

where we write $\zeta_L(s)$ as an Euler product, reflects a function on the ideles

$$(\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \times \prod_{\mathcal{P}}{}' L_{\mathcal{P}}^\times$$

We also can write the Dedekind-zeta function as a product of L-functions twisted by characters of a cyclotomic extension, which can be understood as a product of a function of the uniformiser at each completion. Tate showed this can prove desirable properties of the completed zeta function or twisted L-functions; understanding these factors as a product of local factors and local integrals and, importantly, using Fourier analysis gives rise to an elegant mimicry of Riemann's and Hecke's techniques for studying the zeta function. The focus will be on the finite primes but there are analogous results (following similar proofs) for the infinite primes. The one-dimensional case is quite rich, so further specialisation will be made, here I mostly follow Tate's thesis [Tat50].

2.3.1 Fourier Analysis

Fact. *A locally compact topological topological group G has a left-translation invariant Haar measure $\mu_G = \int dx$. Moreover, this is unique up to scaling.*

This is the first crucial result giving access to Fourier analysis on the locally compact topological group $K^\times$, we then need to ensure that we have the usual duality results to allow for Fourier inversion. Now we set up the other components to define a Fourier transform; the functions we integrate and the analogue of the exponential.

Definition. *Say $f : K \to \mathbb{C}$ is a Schwartz function if it is locally constant with compact support; there is non-negative integer m such that supp$(f) \subset (\pi_K^m)$ and f is constant on cosets of (π_K^m). The complex vector space of Schwartz-Bruhat functions is denoted $S(K)$.*

Note the support of f is the closure of $\{x \in K : f(x) \neq 0\}$. These imitate the functions that decay very quickly (to allow for a convergent Fourier transform). The property that f is locally constant is really the analogue of smoothness of functions over $\mathbb{R}$ or $\mathbb{C}$, note that f is continuous when $\mathbb{C}$ is equipped with discrete topology so smoothness is a much stronger property (this will be important in section 3). Compact support is useful for the Fourier transform, in this sense the two properties defining f are dual - the Fourier transform will be an isomorphism of $S(K)$. In our case, $S(K)$ has a nice basis given by the characteristic functions $\mathbb{1}_{a+\pi_K^m \mathcal{O}_K}$ for $a \in K$, $m \in \mathbb{Z}$.

Definition. *An additive character is a non-trivial continuous unitary homomorphism $\psi : K \to \mathbb{C}^\times$.*

Continuous homomorphism is a strong enough property to ensure its image is in S^1 by compactness of $\mathcal{O}_K$ and its translates, so we don't quite have the freedom as we did with quasi-characters. There is a canonical additive character given by the composition

$$K \xrightarrow{\mathrm{Tr}_{K/\mathbb{Q}_p}} \mathbb{Q}_p \twoheadrightarrow \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \xrightarrow{e^{2\pi i(\cdot)}} S^1 \subset \mathbb{C}^\times$$

which will play the role of the exponential in the Fourier transform, though for now we just need the existence of an additive character. Note the kernel of this map is the inverse different $D_{K/\mathbb{Q}_P}^{-1}$.

Definition. *The Fourier transform of $f \in S(K)$ with respect to (ψ, μ_K) is*

$$\hat{f}(y) = \int_K f(x)\psi(xy)dx$$

For K we normalise the Haar measure so that $\mu_K(\mathcal{O}_K) = N(D_K)^{-\frac{1}{2}}$ where D_K is the different ideal. We can identify K with its dual via $x \mapsto (y \mapsto \psi(xy))$, a general and rigorous approach is taken by Taibleson [Tai75], which in our case, gives us a Fourier inversion result by checking the Fourier transform of characteristic functions as above is in $S(K)$. It also allows us to adjust the canonical additive character so that it is unramified.

Lemma 2.14. *With the canonical additive character and normalised Haar measure μ_K as above, we have the Fourier inversion formula $\hat{\hat{f}}(y) = f(-y)$. This holds more generally for any additive character ψ, and a corresponding (unique) normalised Haar measure dx called the self-dual Haar measure w.r.t ψ.*

Note the self-dual Haar measure is defined so that the Fourier transform defines an isometry of $S(K)$ with respect to the L^2-norm. This result is key in Tate's proof of a version of the Poisson summation formula over the adeles.

Finally, we find the translation invariant Haar measure on $K^\times$. It is not hard to spot in this case, but we can do this more generally by using the fact that dx is a translation invariant differential on the formal additive group. For a formal group $\mathcal{F}$ we define an isomorphism of formal groups given by a logarithm $\log_{\mathcal{F}} : \mathcal{F} \to \mathbb{G}_a$ for which we have a commuting diagram and induced diagram on differentials

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\log_{\mathcal{F}}} & \mathbb{G}_a \\ {\scriptstyle\tau_x}\downarrow & & \downarrow{\scriptstyle\tau_{\log x}} \\ \mathcal{F} & \xrightarrow{\log_{\mathcal{F}}} & \mathbb{G}_a \end{array} \qquad \begin{array}{ccc} \Omega_{\mathcal{F}} & \xleftarrow{\log_{\mathcal{F}}^*} & \Omega_{\mathbb{G}_a} \\ {\scriptstyle\tau_x^*}\uparrow & & \uparrow{\scriptstyle\tau_{\log x}^*} \\ \Omega_{\mathcal{F}} & \xleftarrow{\log_{\mathcal{F}}^*} & \Omega_{\mathbb{G}_a} \end{array}$$

where τ is the translation map. For example, $\log_{\mathbb{G}_m}(X) = \log(1+X)$, so pulling back dx we get $\frac{dx}{1+x}$ is translation invariant in $\Omega_{\mathbb{G}_m}$. Thus, we have a translation invariant Haar measure on $K^\times$ given by $d^\times x = \frac{dx}{|x|_K}$. Then define

$$d^\times x = (1 - q^{-1})^{-1} \frac{dx}{|x|_K}$$

on $K^\times$, this is so that $\mu_{K^\times}(\mathcal{O}_K^\times) = 1$, and so in the global setting we can define the Haar measure on the ideles to be the product of all the local ones (which converges after this normalisation). For calculations, it is easier to work with the normalised measures and canonical additive character, but we will see that our choice does matter in certain contexts.

Definition. *Let $\chi : K^\times \to \mathbb{C}^\times$ be a quasi-character. The local zeta integral is*

$$\zeta(f, \chi) = \int_{K^\times} f(x)\chi(x) d^\times x$$

for $f \in S(K)$. This is absolutely convergent for Re$(s) > 0$.

e.g. For $K = \mathbb{Q}_p$ and $s \in \mathbb{C}$ we have

$$\zeta(\mathbb{1}_{\mathbb{Z}_p}, |\cdot|_K^s) = (1 - p^{-s})^{-1}$$

by filtering $\mathbb{Z}_p$ over the family of powers of $p\mathbb{Z}_p$ noting that $\mu_{\mathbb{Q}_p}(p^n\mathbb{Z}_p) = p^{-n}$. The product over all primes retrieves the usual zeta function, justifying its name, and the gamma factor in the completed zeta function comes from an analogous calculation for the archimedean places. Tate showed (with the normalised Haar measure on $K^\times$) the following functional equation;

Theorem 2.15. (*Local functional equation*) *Let $\chi^\vee = \chi^{-1}|\cdot|_K$ be the twisted dual of χ, ψ an additive character of K and dx a translation invariant Haar measure on K. Then we have the functional equation*

$$\zeta(\hat{f}, \chi^\vee)L(\chi^\vee)^{-1} = \epsilon(\chi, \psi, dx)\zeta(f, \chi)L(\chi)^{-1}$$

for some meromorphic function $L : \hat{K}^\times \to \mathbb{C}^\times$ and holomorphic function $\epsilon : \hat{K}^\times \to \mathbb{C}^\times$. From now we take this as an equality of their meromorphic continuations (following results hold by the identity principle).

Remarks 2.16.

(i) It is often useful to twist the character by ω^s, the integral still converges as we are working with functions with compact support. A cleaner display is

$$z(\hat{f}, \chi^{-1}, 1-s)L(\chi^{-1}, 1-s)^{-1} = \epsilon(\chi, \psi, dx, s)z(f, \chi, s)L(\chi, s)^{-1}$$

where $z(f, \chi, s) = \zeta(f, \chi\omega^s)$ and $L(\chi, s) = L(\chi\omega^s)$.

(ii) Define

$$z_0(f, \chi, s) = \int_{K^\times} (f(x) - f(\pi_K^{-1}x))\chi(x)\omega^s(x)d^\times x$$

Then by splitting the integrand and rescaling by πx, we have $z_0(f, \chi, s) = z(f, \chi, s)L(\chi, s)^{-1}$ so the functional equation is

$$z_0(\hat{f}, \chi^{-1}, 1-s) = \epsilon(\chi, \psi, dx, s)z_0(f, \chi, s)$$

Note for $f = \mathbb{1}_{\mathcal{O}_K}$ we have $f(x) - f(\pi_K^{-1}) = \mathbb{1}_{\mathcal{O}_K^\times}$ so

$$z_0(\mathbb{1}_{\mathcal{O}_K}, \chi, s) = \int_{\mathcal{O}_K^\times} \chi(x)dx$$

is independent of s.

(iii) The local zeta integrals are independent of the choice additive character and Haar measure on K (as scaling K is translation on $K^\times$), so we can use the preferred ones from before, but the ϵ-factor inherits the dependency on these from the Fourier transform.

These L and ϵ-factors are one-dimensional, we now define them generally for Weil-Deligne representations.

2.3.2 L-factors

The local L-factors are easy to define, and they build global L-functions such as the Dedekind zeta function. For (ρ, N) a Weil-Deligne representation on V, let

$$V^{I_K} = \{v \in V \mid \rho(x)v = v \ \forall x \in I_K\} = \bigcap_{x \in I_K} \ker(\rho - 1)(x)$$

and $V_N = \ker N$. Then $V_N^{I_K} = V_N \cap V^{I_K}$ is an invariant subspace by 2.5 (ii) and the fact that I_K is normal in W_K. Let $\chi : W_K \to \mathbb{C}^\times$ be a quasi-character, passing to its abelianisation and using local Artin reciprocity we can view this as a character on $K^\times$.

Definition. *The L-factor associated to (ρ, N) is*

$$L(\rho, N, s) = det(1 - q^{-s}\Phi|_{V_N^{I_K}})^{-1}$$

meromorphic in s.
In particular, the local L-factor is

$$L(\chi, s) = L(\chi\omega^s) = (1 - q^{-s}\chi(\pi_K))^{-1}$$

if χ is unramified (and 1 otherwise).

e.g. for the trivial character

$$L(1, s) = (1 - q^{-s})^{-1} = \zeta(\mathbb{1}_{\mathcal{O}_K}, |\cdot|_K^s)$$

and the product over all primes gives the Dedekind zeta function. In general we can piece up these local factors to prove properties of naturally arising twisted L-functions in the global setting. An important observation in this example is the pole at $s = 0$ which is not true for any non-trivial character, this gives us a way to distinguish the trivial character just as in the global setting in the proof of Dirichlet's theorem on arithmetic progression, though here we will use it for injectivity in the correspondence.

We will be defining several functions of representations of a group G with value in some abelian group A, as before it is useful to know when they extend to homomorphism of the group of virtual representations $R(G)$. A necessary and sufficient condition is that the function is additive, in the following sense.

Definition. *A function $\lambda : Rep(G) \to A$ extends to a homomorphism on $R(G)$ iff it is additive; that is for any exact sequence of representations $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ we have $\lambda(V) = \lambda(V')\lambda(V'')$.*
For L/K finite, a function $\lambda(L, \cdot) : Rep(W_L) \to A$ is inductive over K if it is additive, and for any finite tower $M/L/K$ we have $\lambda(M, [\cdot]) = \lambda(L, Ind_{M/L}[\cdot])$ on $R(W_M)$.

The map $\rho \mapsto L(\rho, s)$, has the nice (obvious) following structural property, which, on those with trivial nilpotent operators, is additive in the above sense.

Lemma 2.17. *For Weil-Deligne representations ρ, ρ'*

$$L(\rho \oplus \rho', s) = L(\rho, s)L(\rho', s)$$

In fact, $L(\cdot, s)$ is additive on short exact sequences of representations of W_K, so we have a homomorphism from $R(W_K)$ to the group of meromorphic functions on $\mathbb{C}^\times$ given by $(\rho, 0) \mapsto L(\rho, s)$.

Recall here we identify a representation ρ of W_K with the Weil-Deligne representation $(\rho, 0)$. Since by 2.11 $R(W_K) = R^0(W_K) + \mathbb{Z} \cdot [1]$ where R^0 are the virtual representations of degree 0, we can understand L-factors by studying the abelian case.

For an ℓ-adic representation ρ_ℓ on V it is natural to define the L-factor as $\det(1 - q^{-s}\Phi|_{V^{I_K}})^{-1}$ after embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, as it is the reciprocal of the characteristic polynomial of the Frobenius element (evaluated at q^{-s}). This coincides with the L-factor of the associated Weil-Deligne representation (ρ, N) where

$$\rho(g) = \rho_\ell(g) \exp(-t_\ell(x) N_\ell)$$

for $g = \Phi^m x$, $x \in I_K$ and $N = 1_{\mathbb{C}} \otimes N_\ell$ as described by the monodromy theorem. Indeed, I_K has finite image under ρ so we have V^{I_K} is contained in $\ker N_\ell$, hence, the I_K invariants of $\bar{V} = \mathbb{C} \otimes V$ in the kernel of N is precisely $\mathbb{C} \otimes V^{I_K}$, thus

$$\rho(\Phi)|_{\bar{V}_N^{I_K}} = 1_{\mathbb{C}} \otimes \rho_\ell(\Phi)|_{V^{I_K}}$$

2.3.3 Conductors

Generally, it is cleaner to associate various numbers of geometric objects, such as the conductor, using the Tate module and proceeding via Weil-Deligne representations. For the one-dimensional case, we can use local reciprocity to understand conductors well and rephrase it in a representation theoretic flavour to generalise. Without this, a higher non-abelian reciprocity would need to be sought after to write them explicitly which poses a challenge apparent in the explicitness of ϵ-factors.

Fact. *L/K an abelian extension, $G = Gal(L/K)$, then (restriction of the) local Artin map sends the m^{th} unit group $U^m = 1 + \pi_K^m \mathcal{O}_K$ to the m^{th} higher ramification group G^m in upper numbering.*

In terms of lower numbering $G^{\phi(s)} = G_s$ where

$$\phi(s) = \phi_{L/K}(s) = \int_0^s [G_0 : G_t]^{-1} dt$$

The upper numbering works well with quotients; the image of G^t in G/H is $(G/H)^t$ (Herbrand's theorem). So we have a natural filtration of $K^\times$ by unit groups which maps to the filtration of the inertia by ramification groups as intuited before. For any abelian extension, we define the conductor of a representation by seeing how deep we have to go in this filtration before it acts trivially.

Now let L/K be any Galois extension, $\chi : \mathrm{Gal}(L/K) \to \mathbb{C}^\times$ a one-dimensional character, then $\ker \chi$ is a finite index subgroup with cyclic quotient by factoring through the abelianisation (and any finite subgroup of $\mathbb{C}^\times$ is cyclic). The conductor $f(\chi)$ of χ is smallest integer m such that

$$\mathrm{res}_{\ker \chi/K} \circ \theta(U^m) = 1$$

where ker $\chi \subseteq L$ by the Galois correspondence. The fact implies

$$f(\chi) = \phi(c_\chi) + 1$$

where c_χ is the largest integer for which G_c acts non-trivially. In particular, for the trivial character, its conductor measures how deeply ramified the extension L/K is. Let

$$\chi(G_s) = |G_s|^{-1} \sum_{x \in G_s} \chi(x)$$

be the average of χ on G_s.

Lemma 2.18. $f(\chi) = \sum_{s=0}^{\infty} [G_0 : G_s]^{-1}(1 - \chi(G_s))$.

Proof. This follows by orthogonality of characters, so the sum is only up to c_χ, and by transitivity of ϕ on extensions noting $c_1 = \phi_{L/\ker\chi}(c_\chi)$ by Herbrand's theorem. ☐

For a character χ of $\mathrm{Gal}(L/K)$ of a representation ρ on V, we can define its conductor using 2.18, replacing 1 with $\chi(1) = \dim V$ and noting that the average $\chi(G_s) = \dim V^{G_s}$ where, as in the proof of Maschke's theorem, the average of an operator over a group projects to the subspace invariant under the group action;

$$c(\rho) = c(\chi_\rho) = \sum_{s=0}^{\infty} [G_0 : G_s]^{-1}(\dim V - \dim V^{G_s}) = \int_{-1}^{\infty} \mathrm{codim}_V V^{G^t} dt$$

We can mimic this for a representation ρ of W_K; the inertia acts with finite image, so we can measure how deep we have to go in this filtration before we get a trivial action. For any finite Galois extension L/K^{ur} for which $\mathrm{Gal}(\bar{K}/L)$ acts trivially, define the conductor of ρ as

$$c(\rho) = \sum_{s=0}^{\infty} [G_0 : G_s]^{-1}(\dim V - \dim V^{G_s})$$

where the ramification groups are those of $\mathrm{Gal}(L/K^{\mathrm{ur}})$. By Herbrand's theorem define $G^t = \{\sigma \in G_K : \sigma|_L \in G^t_{L/K} \text{ for all finite } L/K\}$ as an inverse limit, then we get a cleaner definition:

Definition. *For a representation ρ of W_K with character χ_ρ, its conductor is*

$$c(\rho) = c(\chi_\rho) = \int_{-1}^{\infty} \mathit{codim}_V V^{G^t} dt$$

As with the conductor of an extension of K, $c(\rho) = 0$ iff ρ is unramified. Having defined the conductor of characters, the following characterises it for representations of W_K by 2.11.

Lemma 2.19. *We have the following characterising properties:*

(i) $c(\cdot)$ *is additive on short exact sequences.*

(ii) L/K *finite with relative discriminant* $N(D_{L/K}) = (\pi_K^d)$ *then*

$$c(Ind_{L/K}\rho) = d \cdot dim\ \rho + f(L/K)c(\rho)$$

where $f(L/K)$ *is the conductor of the trivial character on* $Gal(L/K)$*.*

The proof is standard computation similar to before (see [Ser79]). In line with the proof of the L-factor of an ℓ-adic representation coinciding with that of its associated Weil-Deligne representation, we can define the conductor of the associated operator N on V to be

$$c(N) = \mathrm{codim}_{V^{I_K}} V_N^{I_K}$$

this is clearly dependent on ρ but the notation is suppressed (reserved for Weil-Deligne representations).

Definition. *The conductor of a Weil-Deligne representation* (ρ, N) *is* $c(\rho, N) = c(\rho) + c(N)$*.*

We immediately see that $c(\rho, N) = c(\rho)$ if $N = 0$, so the conductor of an irreducible Weil-Deligne representation is just that of the associated representation which is consistent with the identification preceding 2.8. In fact, we can retrieve the conductor of N from the L-factors of the representation and its associate irreducible one by thinking of L-functions as (reciprocals) of characteristic polynomials evaluated at q^{-s}, this gives:

Lemma 2.20. *The degree of* $\frac{L(\rho,N,s)}{L(\rho,0,s)}$ *as a polynomial in* q^{-s} *is* $c(N)$*.*

Thus, the additive and inductive properties of L-functions, with the other properties in 2.19, carry over to conductors of Weil-Deligne representations.

Corollary 2.21.

(i) $c(\rho \oplus \rho', N \oplus N') = c(\rho, N) + c(\rho', N')$*. Moreover, it defines a homomorphism on* $R(W_K)$*.*

(ii) $c(Ind_{L/K}\rho, N) = d \cdot dim\ \rho + f(L/K)c(\rho, N)$ *with notation as in 2.19.*

2.3.4 ϵ-factors

ϵ-factors are harder to work out due to their dependencies, mainly on characters for which we need to understand them well, but we can characterise them in terms of certain symmetry properties. It is important to have a good idea as to how they behave in the abelian case which Tate worked out explicitly.

Proposition 2.22. *With notation as before,* χ *a one-dimensional character of* W_K*,* $n(\psi)$ *the largest integer such that* $\psi(\pi_K^{-n(\psi)}) = 1$ *and* $a = \pi_K^{n(\psi)+c(\chi)}$*, then*

(i) $\epsilon(\chi, \psi(xy), rdx) = r\chi^\vee(y)\epsilon(\chi, \psi, dx)$ *for* $r > 0$*,* $y \in K^\times$

(ii) $\epsilon(\chi,\psi,dx) = \begin{cases} (\chi^\vee)^{-1}(a)\ \mathit{Vol}(\mathcal{O}_K, dx), & \text{if } \chi \text{ unramified} \\ \int_{a^{-1}\mathcal{O}_K^\times} \chi^{-1}(x)\psi(x)dx, & \text{if } \chi \text{ ramified} \end{cases}$

Remark 2.23. *Identifying K with its dual, any additive character is given by $\psi_y(x) = \psi(xy)$, and we can rescale the Haar measure so that it is self dual to ψ. Thus by 2.22 (i) we can compute the ϵ-factor for our preferred additive character. Moreover, by 2.22 (ii) and (iii) we have the useful relationship*

$$\epsilon(\chi,\psi,dx,s+1) = q^{-(n(\psi)+c(\chi))}\epsilon(\chi,\psi,dx,s)$$

The dx is usually dropped if we assume its the dual Haar measure.

Proof. The definition of the ϵ-factor comes from the local functional equation, so we can exploit the symmetries arising in various components of 2.15 and have freedom in choice of f. Full details of the computation are in Tate's thesis.

2.16 (iii) immediately shows $\epsilon(\chi,\psi,rdx) = r\epsilon(\chi,\psi,dx)$. The transformation of the additive character also follows from 2.16 (iii) and that if $\hat{f}_\psi$ is the Fourier transform with respect to ψ then $\hat{f}_{\psi_y}(x) = \hat{f}_\psi(xy)$, it is easiest to compute the ratios of z_0 (for these two transforms) as in 2.16 (ii). The unramified case is computed using 2.16 (ii) with $f = \mathbb{1}_{\mathcal{O}_K}$, and the ramified case follows similarly with $f = \omega^{-1}(\mathbb{1}_{\mathcal{O}_K} - \mathbb{1}_{\pi_K\mathcal{O}_K}) = \omega^{-1}\mathbb{1}_{\mathcal{O}_K^\times}$. ∎

For a one-dimensional character of W_K we identify it with a character on $K^\times$ using Artin reciprocity, so it is not obvious how to generalise the definition of the ϵ-factor for a representation ρ of W_K. Langlands proved (with a short proof by Deligne [Del73b]):

Theorem 2.24. *Let $\rho \in \mathit{Rep}(W_K)$. There is a unique function $\epsilon(\rho,\psi,dx)$ characterised by*

(i) *For $\rho = \chi$ one-dimensional, $\epsilon(\chi,\psi,dx)$ is determined as in 2.15*

(ii) *$\epsilon(\cdot,\psi,dx)$ is additive on $\mathit{Rep}(W_K)$ and so defines a homomorphism on $R(W_K)$*

(iii) *$\epsilon([\cdot],\psi\circ Tr,dx)$ is inductive in degree 0; for any finite tower $M/L/K$, Haar measures dx_L on L and dx_M on M, then $\forall\ [\rho] \in R^0(W_K)$*

$$\epsilon(\mathit{Ind}_{M/L}[\rho],\psi\circ Tr_{L/K},dx_L) = \epsilon([\rho],\psi\circ Tr_{M/K},dx_M)$$

Thus, $\epsilon(\mathit{Ind}_{L/K}[\rho],\psi,dx) = \epsilon([\rho],\psi\circ Tr_{L/K},dx_L)\left(\frac{\epsilon(\mathit{Ind}_{L/K}[1_L],\psi,dx)}{\epsilon([1_L],\psi\circ Tr_{L/K},dx_L)}\right)^{\dim\rho}$, which using 2.11 allows us to write this in terms of one-dimensional ϵ-factors.

Overall, additivity of L and ϵ-factors tells us we only need to work with indecomposable representations. Using these structural properties and the one-dimensional case, we can further deduce properties that are more extrinsic in flavour.

Corollary 2.25.

(i) $\epsilon(\rho, \psi_a, rdx) = r^{dim\ \rho}|a|_K^{-dim\ \rho}(det\ \rho(a))\epsilon(\rho, \psi, dx)$

(ii) $\epsilon(\rho, \psi, dx, s+1) = q^{-(n(\psi)(dim\ \rho)+c(\rho))}\epsilon(\rho, \psi, dx, s)$

Before we define the ϵ-factor of a Weil-Deligne representation (ρ, N), we will have to understand one more number. We saw we could think of the conductor of N as the degree of $\frac{L(\rho,N,s)}{L(\rho,0,s)}$, we can also work out the ratio of the coefficients of the leading terms, using 2.20, multiplicativity of the determinant, the definitions of $c(N)$ and the L-function, we have

Lemma 2.26. *The leading coefficient of* $\frac{L(\rho,N,s)}{L(\rho,0,s)}$ *is*

$$\lim_{s\to-\infty} q^{-sc(N)}\frac{L(\rho, N, s)}{L(\rho, 0, s)} = det(-\Phi|_{V^{I_K}/V_N^{I_K}})$$

The first expression as a limit shows this number inherits the additive and inductive properties from the L-function and the conductor. Write $\rho' = (\rho, N)$ for the Weil-Deligne representation.

Definition. *The ϵ-factor of ρ' is*

$$\epsilon(\rho', \psi, dx) = det(-\Phi|_{V^{I_K}/V_N^{I_K}})\epsilon(\rho, \psi, dx)$$

with twist

$$\epsilon(\rho', \psi, dx, s) = q^{-s(n(\psi)(dim\ \rho')+c(\rho'))}\epsilon(\rho', \psi, dx)$$

Breaking this up into its two factors, we see that 2.24 holds if we replace ρ with ρ' and W_K with W_K'.

We can compute these numbers for the special representation $(sp_m(\rho), N)$ in terms of the irreducible representation ρ of W_K. Let δ be the valuation of the different ideal $D_{K/\mathbb{Q}_p} = (\pi_K^\delta)$ and take the canonical additive character with self dual Haar measure. Then

$$L(sp_m(\rho), N, s) = L(\rho, 0, s+m-1)$$

$$c(sp_m(\rho), N) = \begin{cases} m-1, & \text{if } \rho \text{ unramified} \\ mc(\rho), & \text{if } \rho \text{ ramified} \end{cases}$$

$$\epsilon((sp_m, M), \psi, dx) = (-1)^{m-1}q^{-\left(\frac{m\delta}{2}+\binom{m-1}{2}\right)}$$

To see this, first let $(sp_m(\rho), 1\otimes M)$ act on $\mathbb{C}^m \otimes V$ where M acts on $\mathbb{C}^m$ with kernel e_{m-1}. Then the I_K invariants are $\mathbb{C}^m \otimes V^{I_K}$ with $sp_m(\rho)$ acting as $\omega^{1-m}\otimes\rho|_{V^{I_K}}$ on $e_{m-1}\otimes V^{I_K}$, from which the L-factor follows immediately.

If ρ is unramified, then $V^{I_K} = V$ so $c(\rho) = 0$ and $c(M) = m-1$. Note that ρ it factors through W_K/I_K which is abelian so is one-dimensional by Schur's lemma. If ρ is ramified then $V^{I_K} = 0$ (by irreducibility), so $c(N) = 0$. The result of the conductor then follows by additivity as $sp_m(\rho) = \rho \oplus \cdots \oplus \rho(m-1)$ so $c(\rho, N) = mc(\rho) + c(N)$.

For the ϵ-factor, we have $\mathrm{Vol}(\mathcal{O}_K, dx) = q^{-\frac{\delta}{2}}$ and $sp_m = \omega^0 \oplus \cdots \oplus \omega^{m-1}$ is unramified (applying 2.24 (ii), 2.22). The determinant factor follows from $V_N^{I_K} = \mathbb{C}e_{m-1}$ and $\Phi e_i = q^{-i} e_i$ so piecing together gives the result. Note we used $n(\psi) = -\delta$ by the discussion after defining an additive character.

This is incomplete as we haven't worked out what the ϵ-factor attached to $(sp_m(\rho), N)$ is in terms of ρ. It is not hard to work out for ρ unramified; one can show $\epsilon(\rho \otimes \tau, \psi, dx) = \epsilon(\rho, \psi, dx)^{\dim \tau} \det(\tau(\pi_K^{(\dim \rho) n(\psi) + c(\rho)}))$ for τ unramified using its irreducible components are one-dimensional (as in the calculation of the conductor), and noting 2.25 holds if we twist by any one-dimensional unramified character of $K^\times$ (there we substituted $\omega(\Phi) = q^{-1}$). There is, however, a general way using invariant forms to compute the ϵ-factor of an indecomposable Weil-Deligne representation in terms of the associated (2.10) irreducible representation.

2.3.5 Invariant Forms

The idea is that the functional equation gives us a duality relation for the ϵ-factor, so if we are able to find a Hermitian invariant form then the dual can be reinterpreted as a conjugate; we can exploit the theory of unitary operators. This heavily relies on the structure of Weil-Deligne representations from 2.2. Let ρ be a representation of W_K with twisted dual $\rho^\vee = \rho^*\omega$, and ψ an additive character with dual Haar measure $dx = dx_\psi$. The functional equation gives us the following on one-dimensional representations which lifts to general representations by 2.24, we can rewrite this using 2.25.

Lemma 2.27. *The ϵ-factor satisfies the duality relation*

$$\epsilon(\rho, \psi, dx)\epsilon(\rho^\vee, \psi_{-1}, dx) = 1$$

Equivalently,

$$\epsilon(\rho, \psi, dx)\epsilon(\rho^*, \psi, dx) = det\ \rho(-1) q^{(dim\ \rho) n(\psi) + c(\rho)}$$

Let $\rho' = (\rho, N)$ be a representation of W_K' on V. A non-degenerate bilinear (sesquilinear) form on V is ρ'-invariant if for all $g \in W_K'$, $v, w \in V$ we have $\langle \rho'(g)v, \rho'(g)w \rangle = \langle v, w \rangle$. Equivalently, it is ρ-invariant and $\langle Nv, w \rangle = -\langle v, Nw \rangle$. So we have the natural notion of unitary (resp. orthogonal) representations if there is an associated Hermitian (symmetric) invariant form.

Definition. *The root number is $w(\rho', \psi) = \frac{\epsilon(\rho', \psi)}{|\epsilon(\rho', \psi)|}$.*

This is independent of the Haar measure by 2.25. Using this we can now reduce the calculation the ϵ-factor of an indecomposable Weil-Deligne representation $\rho' = (sp_m(\rho), N)$ to the irreducible representation ρ of W_K. Write $\rho = \sigma \otimes \omega^{t + i\theta}$ for $t, \theta \in \mathbb{R}$ and σ of Galois-type (2.3).

Proposition 2.28. *For the canonical additive character ψ with dual Haar measure dx,*

$$\epsilon(\rho', \psi, dx) = w(\rho', \psi) q^{\alpha(1-t-\frac{m}{2})}$$

where

$$\alpha = dim(\rho')n(\psi) + c(\rho')$$

$$w(\rho', \psi) = \begin{cases} (-1)^{m-1}(\sigma\omega^{i\theta})(\Phi)^{mn(\psi)+m-1}, & \text{if } \rho \text{ unramified} \\ w((\rho, 0), \psi)^m, & \text{if } \rho \text{ ramified} \end{cases}$$

Proof. First we define a Hermitian ρ'-invariant form, this is analogous to theory developed to reach Maschke's theorem for complex representations of finite groups. It is quite straightforward as we can write the representation as

$$sp_m \otimes \sigma \otimes \omega^s$$

Now σ has finite image, so we can a Hermitian form invariant under $\sigma \otimes \omega^{i\theta}$. For sp_m we can define

$$\langle v, w \rangle = i^{m-1} \sum_{j=0}^{m-1} v_j w_{m-1-j}$$

which is Hermitian and invariant for $sp_m \otimes \omega^{-(m-1)/2}$. Thus,

$$sp_m \otimes \sigma \otimes \omega^{-(m-1)/2+i\theta}$$

is self dual and we can apply the duality relationship (2.27) to get the square of the ϵ-factor. We have to twist (each ϵ-factor on LHS) back by $\omega^{t+(m-1)/2}$, using 2.25 and 2.27 we get the number α. The root number of ρ ramified follows from $sp_m(\rho) = \rho \oplus \cdots \oplus \rho(m-1)$ and that irreducibility implies the I_K-invariants are trivial (so the determinant factor in the definition of ϵ-factor plays no role). The unramified case follows from 2.25 (twisting by $\rho\omega^{i\theta}$ rather than just ω). $\square$

This all sets up the Galois side of the statement of the correspondence, including the reductions. There is a lot of pleasant theory, and many directions one can go in, this will similarly be true for the representation theoretic side. For the sake of simplicity, the next section will be developed analogously, but the emphasis will be on the parallels as opposed to the motivation or the intricacies of the theory.

3 Representation Theoretic Side

Whilst much of this theory works for locally compact totally disconnected Hausdorff topological groups (called ℓ-groups), $\mathbb{Q}_p$, or more generally, $\mathrm{GL}_n(K)$, is easier to work with as $U_0 = \mathrm{GL}_n(\mathcal{O}_K)$ is a maximal compact open subgroup, unique up to conjugacy. The separated family of compact open subgroups around 1 given by

$$U_m = 1 + \pi_K^m M_n(\mathcal{O}_K)$$

is the important filtration we use to help break down open subgroups into manageable pieces. The main reference for this section is Bernstein and Zelevinsky [BZ77].

3.1 Admissible Representations

Let σ be a representation of $G = \mathrm{GL}_n(K)$ on a complex vector space V, not necessarily of finite dimension.

Definition. *Say σ is*

(i) smooth if $Stab_G(v)$ is open $\forall\ v \in V$,

(ii) admissible if it is smooth and for every open subgroup $U \subseteq G$ the U-invariants V^U is finite dimensional. Such σ is irreducible if the only stable subspaces are 0 and $V (\neq 0)$.

The smoothness condition just says that the representation is smooth (locally constant) in coordinates identifying $\mathrm{Aut}_{\mathbb{C}}(V)$ with $\mathbb{C}^m$. The second condition helps in putting an even stronger notion of continuity of representations, for example we can straightaway deduce a version of Schur's lemma for irreducible admissible representations (that the only G-invariant endomorphisms of V are the scalars). From this we can define the central character on Z - the center of G (scalar multiples of the identity so we can identify with $K^\times$) - as $\sigma(x) = \omega_\sigma(x) 1_V$, which is a quasi-character. It also turns out that irreducible and smooth implies admissible [BZ77], though this is harder. Schur's lemma immediately tells us the smooth irreducible representations of $\mathrm{GL}_1(K)$, but we can say something more general.

Proposition 3.1. *Every finite dimensional smooth irreducible representation of G is one-dimensional and factors through the determinant (i.e. is of the form $\chi \circ det$ for χ a quasi-character).*

Proof. This is another instance of the usefulness of the filtration; the open kernel of a smooth representation contains some U_m, which upon conjugating suitably, ensures that unipotent upper-triangular matrices (and, by normality, the commutator $\mathrm{SL}_n(K)$ of G which is also the kernel of the determinant), is in the kernel. $\square$

So the finite dimensional irreducible admissible representations of G are just characters and very easy to understand. For example if $K = \mathbb{Q}_p$ for p odd, then by the determinant condition of 3.2 we just need to find the smooth characters of $\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times$. The characters of $\mathbb{Z}$ is just $\mathbb{C}$ (continuous with discrete topology) and a character of $\mathbb{Z}_p$ factors through $\mathbb{Z}/p^m\mathbb{Z}$ for m the conductor - this is just a finite cyclic group. The task is now to understand the infinite dimensional ones, but it is not easy to even write down an example. From our filtration we have $U_n/U_{n+1} \cong \mathrm{GL}_n(k)$ and it is useful to first study the representation theory of $\mathrm{GL}_2(F)$ for F finite, which can be done quite explicitly and many techniques can be mimicked well. A more geometric motivation really comes from the desire of a large (or better a transitive) action of G on some space, which we may then understand in terms of the stabiliser of a point and then look at functions on this space as a source of representations. A good condition to put on the stabiliser is compactness, combining this with an algebraic condition essentially means we look for P such that $P\backslash G$ is projective.

Before that, lets look at a little more theory and a different perspective on obtaining representations. Another application of smoothness is duality, let V^* be the dual representation of σ. Since the above filtration is a fundamental system of neighbourhoods of 1, we can write

$$V = \bigcup_U V^U$$

over open compact subgroups, this is also sufficient to ensure smoothness and can prove a version of Schur's lemma too, it also gives:

Lemma 3.2. *Define $V^\vee = \{\lambda \in V^* : Stab_G(\lambda) \text{ is open}\}$ with induced representation $\sigma^\vee$. Then σ is admissible (resp. irreducible) iff $\sigma^\vee$ is admissible (irreducible), in which case V is isomorphic to $V^{\vee\vee}$.*

3.1.1 Hecke Modules

This will give us our first source, but also allows us to define a trace - in this sense it is quite hands-on. We saw the one-dimensional case of smooth functions on K in 2.3.1, this generalises easily to G; let $S(G)$ be the space of functions $f : G \to \mathbb{C}$ that are locally constant and have compact support; the Schwartz functions. Take a left-invariant Haar measure μ on G so we can define $\int_G f(x)d\mu(x)$. For a compact open $H \subset G$ let $\mathcal{H}(G, K) \subset S(G)$ be the subset of left and right H-invariant Schwartz functions. This is in fact an associative algebra via the convolution product $\mathcal{H}(G, H) \times \mathcal{H}(G, H) \to \mathcal{H}(G, H)$,

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

with two-sided identity $e_H = \frac{1}{\mu(H)} \mathbb{1}_H = \frac{1}{\mathrm{Vol}(H)} \mathbb{1}_H$. We can understand these in terms of double cosets, as each $f \in \mathcal{H}(G, H)$ is a function on $H\backslash G/H$ with finite support, so as vector space it has a basis consisting of characteristic functions of double cosets - a proof of this comes from compactness of the support of f and that f is constant on $K_x x K_x$ for some compact open subgroup K_x as f locally constant (so continuous w.r.t the discrete topology). So we have an associative algebra given by the direct limit

$$S(G) = \bigcup_H \mathcal{H}(G, H)$$

called the Hecke algebra $\mathcal{H}(G)$ (though the identity changes as we range over compact open subgroups H so it is not unital).

For a compact open $H' \subset H$, the inclusion doesn't send the identity to the identity, but we do have the algebraic understandings

$$e_H = e_{H'} e_H$$

$$\mathcal{H}(G, H) = e_H \mathcal{H}(G) e_H$$

We can define a representation of $\mathcal{H}(G)$ just as $\mathbb{C}$-linear map to $\mathrm{Aut}_{\mathbb{C}}(V)$ respecting multiplication, but its easier to express in the language of modules, intuitively the Hecke algebra can be thought of as to play the role of $\mathbb{C}[A]$ for a finite group A.

Definition. *A smooth representation V of G is a $\mathcal{H}(G)$-module via*

$$fv = \int_G f(g) g(v) d\mu(g)$$

We can usually interpret this integral using vectors, here it will be useful to define the matrix coefficient for $\lambda \in V^\vee$ and $v \in V$ by

$$\pi : G \to \mathbb{C}, \ \pi(g) = \lambda(gv)$$

where $\pi = \pi_{\lambda, v}$ this can be thought of as a generalisation of retrieving the coefficient from a matrix using an inner product $\langle g(v), w \rangle$ if we choose v and w to be standard basis vectors of a finite dimensional space. It also makes it easier to see what it is meant for a representation to be smooth in coordinates - its simply the smoothness condition on matrix coefficients. The integral itself can now be thought of in coordinates as

$$\lambda(fv) = \int_G f(g) \pi(g) d\mu(g)$$

These induced representations are also non-degenerate in the sense that $\mathcal{H}(G)V = V$. In fact for a non-degenerate $\mathcal{H}(G)$-module V we can define a G-action via the isomorphism

$$\mathcal{H}(G) \otimes_{\mathcal{H}(G)} V \cong V$$

where $ge_H = \frac{1}{\mu(H)} \mathbb{1}_{gH}$ which gives [Ca 0]:

Proposition 3.3. *The above induces an equivalence of categories*

$$\{\text{smooth representations of } G\} \longrightarrow \{\text{non-degenerate } \mathcal{H}(G)\text{-modules}\}$$

Under this equivalence, an admissible representation is sent to one for which every element of $\mathcal{H}(G)$ has finite rank, as for $h \in \mathcal{H}(G, H)$

$$hV = he_H V = V^H$$

This gives a sense to why Hecke algebras are important; we can define the character χ_σ of an admissible representation σ by taking the trace of the associated linear operator. These satisfy the usual facts that the characters of non-isomorphic representations are linearly independent.

3.2 Bernstein-Zelevinsky Classification

We now look at the building blocks for irreducible admissible representations. The supercuspidal representations don't quite give all of them in that form but the rest can be obtained using parabolic induction, in fact it is easier to get a feel as to their fundamental building property this way but there is a simple definition using matrix coefficients.

Definition 3.4. *A smooth representation V is supercuspidal if every associated matrix coefficient $\pi_{\lambda,v}$ has compact support modulo Z. That is for every $\lambda \in V^\vee$, $v \in V$ there is a compact subset $H_{\lambda,v}$ of G for which $\pi_{\lambda,v}$ vanishes outside $H_{\lambda,v}Z$.*

If Z were compact this would just be a compactly supported representation and in fact would just look like induction from Z with a strong uniform condition.

3.2.1 Parabolic Induction

For finite groups, restriction and induction are right and left adjoints, which is not true in general, so there are two definitions of induction.

Definition. *For a closed subgroup $H \leq G$ and smooth representation σ of H, define*

(i) (smooth induction) $\mathrm{Ind}_H^G(\sigma)$ to be the space of H-equivariant functions $f : G \to V$ such that they are uniformally locally constant - there is an open compact subgroup $N(f)$ on which $f(g) = f(gn)\ \forall n \in N(f)$.

The action of G is by right translation.

(ii) (compact induction) $\mathrm{ind}_H^G(\sigma) \subseteq \mathrm{Ind}_H^G(\sigma)$ consisting of those with compact support modulo H.

There are a few important properties which I will just state.

Lemma 3.5. *Let H be a closed subgroup of G.*

(i) If $H\backslash G$ is compact, then ind = Ind.

(ii) If $H\backslash G$ is compact and σ admissible, then $Ind_H^G(\sigma)$ is admissible.

(iii) (Frobenius Reciprocity) Res is left adjoint to Ind and if further H is open, then Res is right adjoint to ind.

To build representations of G we can consider subgroup of matrices with diagonal blocks in $\mathrm{GL}_{n_i}(K)$ to induct from, but it turns out these lose too much information. The motivation before suggests we should look at parabolics, but it turns out the unipotent subgroup is tricky to work with, so we consider making this act trivially. For a partition $\underline{n} = (n_1, ..., n_r)$ of n consider the parabolic subgroups $P_{\underline{n}}$ consisting of upper-block triangular matrices where the diagonal blocks are in $\mathrm{GL}_{n_i}(K)$. Define the unipotent radical $U_{\underline{n}} \subset P_{\underline{n}}$ to be those with the identity in the diagonal blocks, and Levi factor to be those with with diagonal blocks $\mathrm{GL}_{n_i}(K)$ and zero elsewhere fitting as a quotient in

$$1 \to U_{\underline{n}} \to P_{\underline{n}} \to L_{\underline{n}} \to 1$$

we induct the representations of $P_{\underline{n}}$ trivial on $U_{\underline{n}}$ (inflating from $L_{\underline{n}}$). More generally we consider a parabolic subgroup P of G (contains a maximal Zariski-closed connected solvable subgroup of G; a Borel subgroup) with unipotent radical U and Levi factor

$$L \hookrightarrow P, L \cong P/U$$

Unitary representations are important and connections were made involving admissible irreducible representations to unitary Hilbert space representations; many results come from studying these. It turns out that induction as we defined above does not preserve the representation being unitary, instead there is a normalising factor given by the fact that a left-invariant Haar measure is not quite right-invariant. For $h \in H$ a closed subgroup of G and $\mu : S(H) \to \mathbb{C}$ a right-invariant Haar measure then

$$\left(f \mapsto \int_H f(hx)d\mu(x)\right) = \delta_H(h)\mu$$

Normalised induction is then as before but the equivariant condition is replaced by

$$f(hg) = \delta_H(h)^{\frac{1}{2}} hf(g)$$

One pleasant feature of normalised induction is that it commutes with taking duals as in [BZ77] the dual of $\mathrm{Ind}_H^G(V)$ is $\mathrm{Ind}_H^G(V^\vee \otimes \delta_H^{\frac{1}{2}})$. Summarising, we have the geometric definition:

Definition. *For smooth representations (σ_i, V_i) of $GL_{n_i}(K)$, $\sigma = V = \bigotimes_{i=1}^r V_i$ N-$Ind_{P_{\underline{n}}}^G(V) = \{f : GL_n(K) \to V \mid f(uhg) = \delta_{P_{\underline{n}}}^{\frac{1}{2}}(h)\sigma(h)f(g), u \in U_{\underline{n}}, h \in GL_{\underline{n}}(K), g \in G$ and f smooth$\}$, denoted $\sigma_1 \times \cdots \times \sigma_r$.*

The functor has some nice properties, such as transitivity, left exactness, preserving finite length (of composition series) and admissibility.

We now define the left adjoint to N-Ind (so we have Frobenius reciprocity). For a smooth representation V of a parabolic subgroup P we take the left adjoint to inflation by considering the U-coinvariants

$$V_U = V/\{uv - v : u \in U, v \in V\}$$

as a smooth representation of L. The normalisation by a factor of $\delta^{-\frac{1}{2}}$ gives the Jacquet functor, denoted $J_P^G(V)$ which also has the nice properties of normalised induction. We now define supercuspidal representations, these are the ones that don't arise this way; they are quite true to $\mathrm{GL}_n(K)$ not coming from smaller subgroups.

Definition 3.6. *An irreducible admissible representation σ is supercuspidal if for every proper parabolic subgroup P have $J_P^G(\sigma) = 0$.*

There are some equivalent formulations, the third being more relevant in the Bernstein-Zelevinsky classification of admissible representations.

Theorem 3.7. *For an admissible representation σ, the following are equivalent:*

(i) σ is supercuspidal in the sense of 3.6.

(ii) σ is supercuspidal in the sense of 3.4.

(iii) There is no proper partition of $\underline{n}$ for which σ is a subquotient of a representation $\sigma_1 \times \cdots \times \sigma_r$ for admissible representations σ_i of $GL_{n_i}(K)$.

3.2.2 Classification

Denote $\sigma(s)$ to be the twist by $s \in \mathbb{C}$ of an admissible representation σ so

$$\sigma(s)(g) = |\det(g)|^s \sigma(g)$$

is of the same dimension n. Twisting defines a partial order ($\leq$) on irreducible supercuspidal representations (preserving this property) where $\sigma \leq \sigma'$ iff $\sigma' = \sigma(r)$ for some non-negative integer r. This gives the notion of a segment of length m of irreducible supercuspidals

$$\Delta = \Delta(\sigma, m) = [\sigma, ..., \sigma(m-1)]$$

say two such are linked if their union is a segment but neither is a sub-segment of each other and we get an admissible representation $\sigma(\Delta) = \sigma \times \cdots \times \sigma(m-1)$ of $\mathrm{GL}_{nm}(K)$. Say $\Delta(\sigma_1, m_1)$ precedes $\Delta_2(\sigma_2, m_2)$ if they are linked and $\sigma_1 < \sigma_2$ (in particular, they have the same dimension). Finally, say a the collection of segments $\Delta_1, ..., \Delta_r$ is primitive if for $i < j$, Δ_i does not precede Δ_j, we can always reorder so that this is the case. We can now state the Bernstein-Zelevinsky classification.

Theorem 3.8. *[BZ77]*

(i) $\sigma(\Delta)$ *is reducible iff* $m > 1$*. Moreover it has a unique irreducible subquotient* $Q(\Delta)$ *and subrepresentation* $Z(\Delta)$*.*

(ii) If $\Delta_1, ..., \Delta_r$ *is a primitive collection then* $Q(\Delta_1)\times\cdots\times Q(\Delta_r)$ *has a unique irreducible quotient* $Q(\Delta_1, ..., \Delta_r)$*, independent of primitive reordering.*

(iii) Any irreducible admissible representation σ *is isomorphic to some* $Q(\Delta_1, ..., \Delta_r)$ *unique up to primitive reordering.*

(iv) If $\Delta_1, ..., \Delta_r$ *is a primitive collection then* $Q(\Delta_1)\times\cdots\times Q(\Delta_r)$ *is irreducible iff no two* Δ_i, Δ_j *are linked.*

(v) Write $\Delta^\vee(\sigma, m) = \Delta(\sigma^\vee(1-m), m)$*, then* $Q(\Delta_1, ..., \Delta_r)^\vee = Q(\Delta_1^\vee, ..., \Delta_r^\vee)$*.*

Zelevinsky [Zel80] considered some other admissible representations, (iv) gives an equivalent notion of a generic representation which will be defined in the next section on Whittaker models. It is also a theorem that the $r = 1$ case of (iii) is equivalent to the notion of square integrable representations; those which have square integrable matrix coefficients i.e $\int_{G^0} |\pi_{\lambda,v}|^2 dg$ exists where G^0 is the kernel of the determinant as a multiplicative character. These are quite easy to work with as we have explicit integrals; Whittaker models can be thought of as a generalisation of this to the larger class of generic representations.

We can also see the clear parallels to the Galois side; an irreducible admissible representation is of the form $Q(\Delta_1, \cdots, \Delta_r)$ which comes from the normalised induction (denoted as a product) of square-integrable representations $Q(\Delta_i)$. This mirrors the decomposition of a F-semisimple representation into a direct sum of indecomposable representations. There we saw each such is the special representation of an irreducible representation $sp_m(\rho) = \rho \oplus \cdots \oplus \rho(m-1)$ - a sum of twists looking like a segment - in the same way each $Q(\Delta)$ comes from the normalised induction of twists $\sigma \times \cdots \times \sigma(m-1)$ for an irreducible supercuspidal representation σ, this makes it clear how the reduction theory in both sides match and what is used to define the correspondence.

Example 3.9. (*Steinberg Representation*)
These take care of the irreducible admissible representations that aren't supercuspidal. The diagonal torus is abelian and so its admissible representations can easily be understood; a product of characters.

As an easier case, let's first look at $GL_2(K)$*. Consider a pair of characters* χ_1, χ_2 *of* $K^\times$ *giving a character of* P *defined by*

$$\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto \chi_1(a)\chi_2(d)$$

Note in this case $\delta_T = |\frac{a}{d}|^{\frac{1}{2}}$*. We now follow 3.8, breaking into three cases.*

If $\chi_1 \neq |\cdot|^{\pm 1}\chi_2$ *then normalised induction gives an irreducible representation, which we call a principal series representation, it corresponds quite simply* *to $(\chi_1 \oplus \chi_2, 0)$ on the Galois side. We are left with two special cases that are not irreducible but not hard to understand, overall these along with the supercuspidal representations give us every (infinite dimensional) irreducible admissible representation of $GL_2(K)$.*

In the special case where $\chi_1 = |\cdot|^1 \chi_2$ we get the associated representation (as in 3.8) has a unique irreducible subrepresentation of codimension one, called the special representation of χ_1, χ_2. The unique irreducible quotient is an admissible (one-dimensional) irreducible representation (and so isomorphic to $\chi \circ det$).

The other special case is $\chi_1 = |\cdot|^{-1}\chi_2$ where we get a unique irreducible quotient with one-dimensional kernel, again the special representation of χ_1, χ_2 and irreducible subrepresentation as the quotient in the above case. In particular we can consider $\chi_1 = |\cdot|^{-\frac{1}{2}}$ which gives the representation

$$Q(\Delta(|\cdot|^{-\frac{1}{2}}, 2))$$

$$\sigma|_T = |\cdot|^{-\frac{1}{2}} \otimes |\cdot|^{\frac{1}{2}} = \delta^{-\frac{1}{2}}$$

on the diagonal matrices T.

By 3.8,

$$\sigma(\Delta) = \{smooth\ f : B\backslash GL_2(K) \to \mathbb{C}\}$$

which can be identified with smooth functions on $\mathbb{P}^1(K)$, called the Steinberg representation - this in fact shows that the unique irreducible subrepresentation must in fact be the trivial representation corresponding to the constant functions. Every special representation is then a twist by a character of the Steinberg representation.

This gives all the other special kind of representations of $GL_2(K)$ by twisting with a character of this group, in summary the irreducible admissible representations of $GL_2(K)$ are: irreducible principle series, twists of the Steinberg representation, supercuspidals and then the one dimensional ones factoring through det. I will later compute its L and ϵ-factors which matches with sp_2.

More generally, take the trivial partition $\underline{n} = (1, .., 1)$, the standard Borel $B = P$ given by the upper triangular matrices, diagonal torus $T \subset GL_n(K)$, unipotent upper triangular matrices U, and quasi-character $|\cdot|^{-\frac{n-1}{2}}$. Then

$$|\cdot|^{-\frac{n-1}{2}} \otimes \cdots \otimes |\cdot|^{\frac{n-1}{2}} = \delta^{-\frac{1}{2}}$$

on T. The segment

$$\Delta(|\cdot|^{-\frac{n-1}{2}}, n) = [\ |\cdot|^{-\frac{n-1}{2}}, ..., |\cdot|^{\frac{n-1}{2}}]$$

gives the Steinberg representation

$$|\cdot|^{-\frac{n-1}{2}} \times \cdots \times |\cdot|^{\frac{n-1}{2}} = \{smooth\ f : B\backslash G \to \mathbb{C}\}$$

3.3 L and ϵ-factors

So far the theory seems to have developed quite analogously in spirit but here we have a little incongruity in this flow in defining the factors for pairs of representations, nevertheless we will still have an 'enabler' allowing us to define these factors in an inductive way based off a functional equation. In the Galois side, the key to defining these factors in the abelian case was Tate's local functional equation using Fourier analysis and it was hard to imagine a higher dimensional analogue. Whilst L-factors could be easily generalised, the lack of explicitness of a representation proved to challenge the definition of ϵ-factors. On this side, Whittaker models provide a way to understand a lot of admissible representations explicitly, which makes it easier to perform computations. In particular, they give a way of defining these factors for generic representations, which we can then extend using the Bernstein-Zelevinsky classification. Previously we induced representations of P_n that were trivial on U_n but it is not hard to induce certain one dimensional representations of the unipotent subgroup.

3.3.1 Whittaker Models

Let ψ be a non-trivial additive character of $K^\times$ and view it as a character on U_n by

$$\psi(u_{ij}) = \psi(u_{12} + \cdots u_{n-1,n})$$

An irreducible admissible representation σ is generic (or non-degenerate) if $\mathrm{Hom}_{U_n}(\sigma|_{U_n}, \psi) \neq 0$. The following result characterises generic representation.

Proposition 3.10. *Let σ be an irreducible admissible representation.*

(i) σ is generic iff $\sigma^\vee$ is generic.

(ii) Being generic is independent of choice of ψ.

(iii) An irreducible admissible representation $\tau = Q(\Delta_1, ..., \Delta_r)$ is generic iff no two Δ_i, Δ_j are linked. In this case by 3.11,

$$\tau \cong Q(\Delta_1) \times \cdots \times Q(\Delta_r)$$

(iv) Supercuspidal (and more generally square integrable) representations are generic.

For (σ, V) a generic representation, an element $\lambda \in \mathrm{Hom}_{U_n}(\sigma|_{U_n}, \psi)$ defines for $v \in V$ a smooth function $W_v(g) = \lambda(gv)$ on G. Since we have a representation of U_n we can induce a representation of G in the usual way; ψ-invariant smooth functions.

Definition. *$Ind_{U_n}^G \psi = \{f : G \to \mathbb{C}$ smooth: $f(ug) = \psi(u)f(g), u \in U_n\}$ with G-action given by right translation. A ψ-Whittaker model of (σ, V) is the image $V \to Ind_{U_n}^G \psi, v \mapsto W_v$ denoted $\mathcal{W}(\sigma, \psi)$.*

A ψ-Whittaker functional is a continuous functional $t : V \to \mathbb{C}$ such that $t(\sigma(u)v) = \psi(u)t(v)$ (sometimes just defined as elements of the Whittaker model). Thus $W_{(\cdot)}(1)$ defines a Whittaker functional, conversely for $v \in V$ we have a Whittaker model $\mathcal{W} = \{g \mapsto t(gv)\}$. Shalika showed that Whittaker functionals are unique up to scaling, or equivalently Whittaker models are unique.

3.3.2 Generic case

I will keep this brief but the idea is the same - we can consider local zeta integrals

$$Z(W, s, \phi)) = \int W(g)\phi(g)|\det\ g|^s dg$$

which generates an ideal defining the L-factor and derive a functional equation

$$\frac{Z(\tilde{W}, 1-s, \hat{\phi})}{L(\sigma^\vee, 1-s)} = \epsilon(\sigma, \psi, s)\frac{Z(W, s, \phi)}{L(\sigma, s)}$$

This works well for square integrable representations where we can use matrix coefficients and the methods work well for generic representations using their Whittaker models to explicitly get the integral. In the global case, an automorphic representation has an L-factor with certain lovely properties and [JPS79] considers when would an admissible representation with nice L-factors define an automorphic cuspidal. In order to do so they consider the integrals in 3.11 which can also be modified for pairs of representations [JPS83], important in characterising the bijection uniquely.

Let ψ be a unitary additive character (often with conductor 0) and $\phi \in S(K^n)$ a Schwartz function, the following definitions build on a little theory developed in the case of square integrable representations which mostly follows the same ideas as in the one-dimensional or global case, I will also omit convergence of the integrals, but often they do so on some right half plane.

Definition 3.11. *For an irreducible generic representation σ of $GL_n(K)$ with Whittaker functional $W \in \mathcal{W}(\sigma, \psi)$*

(i) For $0 \le j \le\ n-2$

$$\Psi(W, s; j) = \int\int W\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-j-1} \end{pmatrix}|g|^{s-\frac{n-1}{2}}d^\times g dx$$

where the integration is over $g \in K^\times$ and $x \in K^j$.
Similarly, for $0 \le j \le n-3$

$$\Psi(W, s; j, \phi) = \int\int W\begin{pmatrix} g & & & \\ x & I_j & & \\ y & & 1 & \\ & & & I_{n-j-1} \end{pmatrix}\phi(y)|g|^{s-\frac{n-1}{2}}d^\times g dx dy$$

(ii) Define the L-factor $L(\sigma, s)$ to be the generator of the fractional ideal generated by Ψ (over all choices of W, ϕ) in $\mathbb{C}[q^s, q^{-s}]$ as a rational function $Q^{-1}(q^{-s})$ with $Q(0) = 1$

(iii) *For $\tilde{W}(g) = W(w_n(g^{-1})^t)$ where w_n is the matrix with ones on the anti-diagonal, Fourier transform $\hat{\phi}$ of ϕ (with respect to ψ and its dual Haar measure) modified so $\hat{\phi}'(y) = \hat{\phi}((-1)^{n+j-1}y)$, for $j+k = n-2$ the ϵ-factors are defined by the functional equations*

$$\frac{\Psi(\tilde{W}, 1-s; k)}{L(\sigma^\vee, 1-s)} = \epsilon(\sigma, \psi, s)\frac{\Phi(W, s; j)}{L(\sigma, s)}$$

$$\frac{\Psi(\tilde{W}, 1-s; k-1, \hat{\phi}')}{L(\sigma^\vee, 1-s)} = \epsilon(\sigma, \psi, s)\frac{\Phi(W, s; j, \phi)}{L(\sigma, s)}$$

The proofs refer back to the local zeta integrals by turning the Whittaker functionals into matrix coefficients using a suitable integral. Similarly, for pairs we have:

Definition 3.12. *Consider generic representations σ_i on $GL_{n_i}(K)$ for $i = 1, 2$ and Whittaker functionals $W_1 \in \mathcal{W}(\sigma_1, \psi), W_2 \in \mathcal{W}(\sigma_2, \psi^{-1})$.*

(i) *If $n = n_1 = n_2, \phi \in S(K^n)$ a Schwartz function, define*

$$\Psi(W_1, W_2, s; \phi) = \int_{U_n(K)\backslash G_n(K)} W_1(g)W_2(g)\phi(e_n g)|det\ g|^s dg$$

where e_n is the standard n^{th} basis row vector.

(ii) *If $n_1 > n_2$ and $0 \leq j \leq n_1 - n_2 - 1$, define*

$$\Psi(W_1, W_2, s; j) = \int\int W_1\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n_1-n_2-j}\end{pmatrix} W_2(g)|det\ g|^{s-\frac{n_1-n_2}{2}} dxdg$$

where the integration is over $g \in U_{n_1}(K)\backslash GL_{n_1}(K)$ and $x \in M_{j,n_2}(K)$

(iii) *Define the L-factor $L(\sigma_1 \times \sigma_2, s)$ to be the generator of the fractional ideal generated by Ψ (over all choices of W_i, ϕ) in $\mathbb{C}[q^s, q^{-s}]$ as a rational function of q^{-s} with constant term 1, independent of the choice of ψ.*

(iv) *For the associated central character ω_{σ_2}, the ϵ-factors in each case are defined by the functional equations*

$$\frac{\Psi(\tilde{W}_1, \tilde{W}_2, 1-s; \hat{\phi})}{L(\sigma_1^\vee \times \sigma_2^\vee, 1-s)} = (-1)^{n-1}\omega_{\sigma_2}\epsilon(\sigma_1 \times \sigma_2, \psi, s)\frac{\Phi(W_1, W_2, s; \phi)}{L(\sigma_1 \times \sigma_2, s)}$$

$$\frac{\Psi(w_{n_1,n_2}\tilde{W}_1, \tilde{W}_2, 1-s; i)}{L(\sigma_1^\vee \times \sigma_2^\vee, 1-s)} = (-1)^{n-1}\omega_{\sigma_2}\epsilon(\sigma_1 \times \sigma_2, \psi, s)\frac{\Phi(W_1, W_2, s; j)}{L(\sigma_1 \times \sigma_2, s)}$$

where $i = n_1 - n_2 - 1 - j$ and w_{n_1,n_2} is the matrix with diagonal blocks $I_{n_2}, w_{n_1-n_2}$.

If $n_1 < n_2$ then we just define the factors as if they are switched. We can inductively define the L and ϵ-factors for irreducible admissible representations $\sigma = Q(\Delta_1, ..., \Delta_r)$ as

$$L(\sigma \times \sigma', s) = \prod_i L(Q(\Delta_i) \times \sigma', s)$$

$$\epsilon(\sigma \times \sigma', \psi, s) = \prod_i \epsilon(Q(\Delta_i) \times \sigma', \psi, s)$$

for σ' arbitrary. There are many properties analogous to what we saw before, one of particular use is:

Lemma 3.13. *For a pair of supercuspidal representations σ_1, σ_2 of $GL_n(K)$*

$$L(\sigma_1 \times \sigma_2, s) = \prod L(\chi, s)$$

where the product runs over unramified quasi-characters such that $\chi \otimes \sigma_2^\vee \cong \sigma_1$.

We can return to our example of the Steinberg representation $Q(\Delta(|\cdot|^{-\frac{1}{2}}, 2))$ for $\mathrm{GL}_2(K)$ and compute its L and ϵ-factors. The result is

$$L(\sigma, s) = (1 - q^{-(s+1)})^{-1}$$

$$\epsilon(\sigma, \psi, s) = -q^{-s}$$

where $\sigma = Q(\Delta(|\cdot|^{-\frac{1}{2}}, 2))$. We can immediately recognise these as the corresponding factors for sp_2 if we choose the additive character to be unramified ($n(\psi) = 0$). Using a bit of theory we can compute this by writing the trivial representation as $Q(\Delta(|\cdot|^{\frac{1}{2}}, 1), \Delta(|\cdot|^{-\frac{1}{2}}, 1))$ - so the Steinberg and trivial representations fit into a natural exact sequence - and use the formulae above with 3.13. Often the formulae originally defined in the special case of GL_2 means this representations instead corresponds to a half-twist of sp_2 (where the L-factor is taken to be that corresponding to the first character in the induced representation of the pair).

Explicitly, for $\mathrm{GL}_2(K)$ the one-dimensional admissible representations are matched through local reciprocity, the supercuspidals to the irreducible Weil-Deligne representations, the irreducible principal series representation of two characters to the direct sum of the two, and the Steinberg representations to the special representation sp_2 which deals with all admissible representations by natural compatibility properties.

4 Statement

We are now in a position to state the correspondence, the bijection is canonical via five key properties characterising it uniquely. In the abelian case this is given by the local Artin map which explicitly gives an isomorphism. As sketched in section 3, the $\mathrm{GL}_2(K)$ case is also quite explicit and motivates a lot of the theory and formulation of the correspondence.

4.1 Abelian Case

The one-dimensional Weil-Deligne representations are just complex representations of W_K^{ab} (factoring through abelianisation) continuous with respect to the discrete topology as I_K is compact and totally disconnected, and any one-dimensional nilpotent operator is 0. If it is F-semisimple, then it is irreducible (one-dimensional). By local Artin reciprocity, it therefore can be viewed as an irreducible one-dimensional representation of $K^\times$ continuous with respect to the discrete topology.

Now take an irreducible admissible representation χ of $\mathrm{GL}_1(K) = K^\times$ on V. By irreducibility and commutativity of $K^\times$, the invariants in V of the stabiliser of any $0 \neq v \in V$ is V, hence the representation is finite dimensional and so one-dimensional as any commuting family have a common eigenvector as $\mathbb{C}$ is algebraically closed (this is really just an instance of Schur's lemma). Moreover, ker χ contains $U^m = 1 + \pi_K^m \mathcal{O}_K$ for some m and hence is open. So χ is an irreducible one-dimensional representation of $K^\times$ continuous with respect to the discrete topology.

Local reciprocity identifies any uniformiser to (a lift of) the Frobenius element and the units with the inertia, which exactly pairs up the representations and matches the L and ϵ- factors. The other properties that we will see in the correspondence trivially hold by the specialisations of the theory developed in both sections. Thus, the one-dimensional local Langlands correspondence is local class field theory.

4.2 Correspondence and Reductions

Let A_n be the set of equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(K)$ and G_n the set of n-dimensional F-semisimple Weil-Deligne representations. The correspondence is not an isomorphism but we do have lots of algebraic structure and special building blocks with various associated invariants. The correspondence is a set-theoretic bijection, which is not exciting (by cardinality), the amazing property is that it preserves the structures and invariants which uniquely characterises it. The building of representations on the Galois side required a bit more abstract theory, but this made it quite easy to express combinatorially. In light of this, and local reciprocity, the direction of the map is from the admissible representations of $\mathrm{GL}_n(K)$ as it is easier to check they build the right thing on the Galois side once we have defined where all the objects go.

Theorem 4.1. *There is a unique (canonical) bijection*

$$rec_n : A_n(K) \to G_n(K)$$

independent of additive character ψ of K, characterised by:

(i) (Generalises Class Field Theory) For every $\chi \in A_1(K)$,

$$rec_1(\chi) = \chi \circ Art_K^{-1}$$

(ii) (Compatibility of L and ϵ-factors) For every $\sigma_1 \in A_{n_1}(K),\ \sigma_2 \in A_{n_2}(K)$,

$$L(\sigma_1 \times \sigma_2, s) = L(rec_{n_1}(\sigma_2) \otimes rec_{n_2}(\sigma_1), s)$$

$$\epsilon(\sigma_1 \times \sigma_2, \psi, s) = \epsilon(rec_{n_1}(\sigma_2) \otimes rec_{n_2}(\sigma_1), \psi, s)$$

(iii) (Naturality) For every $\sigma \in A_n(K),\ \chi \in A_1(K)$,

$$rec_n(\sigma\chi) = rec_n(\sigma) \otimes rec_1(\chi)$$

$$rec_n(\sigma^\vee) = rec_n(\sigma)^\vee$$

(iv) For every $\sigma \in A_n(K)$ with associated central character ω_σ,

$$det \circ rec_n(\sigma) = rec_1(\omega_\sigma)$$

Moreover, under rec_n conductors are invariant, square-integrable corresponds to indecomposable, and supercuspidal to irreducible.

The analogous flow in theory of both sides leads to a very nice reduction argument in which the additive properties of L and ϵ-factors play a crucial role. The Bernstein-Zelevinsky classification builds irreducible admissible representations parallel to the decomposition of F-semisimple representations into special representations. This feels quite like a Grothendieck styled argument one sees in geometry.

Let $A_n^0 \subset A_n$ the subset of equivalence classes of supercuspidal representations, and $G_n^0 \subset G_n$ the irreducibles. Then for $\sigma = Q(\Delta_1, ..., \Delta_s)$ where $\Delta_i = [\sigma_i, \sigma_i(m_{i-1})]$ and $\sigma_i \in A_{n_i}^0(K)$, $t = \sum_i m_i n_i$ we define

$$\mathrm{rec}_t(\sigma) = \bigoplus_{i=1}^{s} sp_{m_i}(\mathrm{rec}_{n_i}(\sigma_i))$$

Thus we have a reduction to the supercuspidal case; the existence of a unique bijection $\bar{\mathrm{rec}}_n : A_n^0 \to G_n^0$ characterised by the same properties as rec_n, implies 4.1.

Henniart [Hen93] showed that the ϵ-factor on pairs inductively determines a supercuspidal representation. Hence, by class field theory, uniqueness of 4.1 follows from existence. Since a pole at 0 of the L-factor of a character detects if is trivial, by 3.13 we get injectivity of $\bar{\mathrm{rec}}_n$. Finally, Henniart's numerical local Langlands [Hen83] shows that the conductor determines the same (finite) number of representations on both sides of 4.2, so surjectivity of $\bar{\mathrm{rec}}_n$ follows from injectivity.

4.3 Concluding Remarks

There is a lot of interesting theory in two domains to which there were many analogous analogies which ultimately led to a correspondence of representations with some naturality and preservation of structure. The L and ϵ-factors also ended up being useful in the characterisation in ensuring injectivity and uniqueness of the correspondence, something which seems harder to generalise (particularly ϵ-factors). Langlands proved the local Langlands correspondence for archimedean local fields and it has also been proven for local fields with positive characteristic. The Langlands programme is a series of such predictions and encompasses many flavours; global and local, number fields and functional fields (geometrically flavoured). The case for functional fields was proven by Lafforgue for GL but in general it is hard to even properly state a conjecture.

In the case of number fields, étale cohomology provides a link between algebraic varieties over $\mathbb{Q}$ and geometric Galois representations of $G_{\mathbb{Q}}$. The other links between these two and automorphic representations of GL over the adeles are still conjectural but there's a huge amount of interesting theory. For example, Wiles studied going from Galois representations to modular forms which was an important part of the modularity conjecture. One of the directions from automorphic representations to geometry involves studying Shimura varieties which are used by Harris and Taylor in proving the local Langlands correspondence, I am particularly eager to study this, which despite defining objects in both sides of the correspondence, is still a little mysterious to me hidden in the middle under the theory. It was also used by Deligne to prove certain analytic results conjectured by Ramanujan, which really shows the power of the conjecture as analysis couldn't quite get down to the bound on the size of coefficients appearing in certain modular forms.

Another direction comes from looking for an analogue of Grothendieck's monodromy theorem for $\ell = p$. We saw that eliminating the topology was crucial for creating a family of compatible ℓ-adic representations but in this singular case we naturally would have an interaction in the topology of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_n(\mathbb{Q}_p)$, the difficulty of restyling the proof of the monodromy theorem comes from the fact that ker t_p is no longer unrelated to p. There is a way to untangle this where semistable turns to using 'period rings', and is the study of p-adic Hodge theory. There is still a missing chunk when classifying Weil-Deligne representations as I did not consider when $\ell = p$.

This gives some sense as to the scale of the programme, and it is quite beautiful it can seemingly evolve from quadratic reciprocity - despite that primes talking to each other seems remarkable in its own right, its perhaps now not so surprising there are so many proofs scattered throughout mathematics. The local Langlands correspondence is quite complete as a conjecture but opens many directions and it is exciting that there is a lot of active research going on in most aspects of the Langlands programme and there is much we can learn.

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